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The Dynamic Stability of Elastic Systems

Volume I

(V. V. Bolotin)

Translated by V. I. WEINGARTEN,
K. N. TRIROGOFF and K. D. GALLEGOS

1 NOVEMBER 1962

Prepared for COMMANDER SPACE SYSTEMS DIVISION
UNITED STATES AIR FORCE
Inglewood, California

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(V. V. Bolotin)

Translated by: V. I. Weingarten
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TRANSLATORS' PREFACE TO THE ENGLISH EDITION

Within the past few years, interest in the dynamic stability of elastic systems has increased. This interest is reflected in the appearance of a large number of papers on this subject. Unfortunately, few are in English, the majority of these papers being written either in Russian or German. In addition, there is no English text available which presents the mathematical theory of the subject with its applications. However, such a text is available in Russian. This work, "Dynamic Stability of Elastic Systems," (Gostekhizdat, Moscow, 1956) by V. V. Bolotin, together with its German translation (Veb Deutscher Verlag Der Wissenschaften, Berlin, 1961), is the only comprehensive book on the subject of dynamic stability. To fill this need in American scientific literature, it was decided to undertake the translation of this unique book.

At the time the translation was initiated, the German edition was not yet available, and the book was translated from the Russian. With the appearance of the German edition, the translation was checked and all changes in the German edition were incorporated into the English translation. Footnotes in the Russian edition were eliminated and the German referencing system was adapted for our use. A number of notes were made by the translators in order to bring the American edition of the text up to date. References to new works published in the United States and an additional Russian reference were added.

The translators would also like to thank Dr. Paul Seide, Dr. Robert Cooper, and Dr. John Yao of Aerospace Corporation for reading the translation.

El Segundo, California, November 1962 V.I.W., K.N.T., K.D.G.

PREFACE TO THE GERMAN EDITION

Four years have passed since the appearance of the Russian edition of the present book. During this time a series of works on dynamic stability were published, containing interesting results. In addition, the general theory was applied to a new class of problems whose analysis appeared to be of a mathematical nature which fell within the narrow limits of the theory of dynamic stability. All of the above are considered in the German edition of the book. References are included of some works published thru 1956, which were unknown to me when the book was first published.

G. Schmidt, from the Institute for Applied Mathematics and Mechanics, German Academy of Sciences, Berlin, prepared the excellent translation and contributed corrections in some places. The translation was also reviewed at the Institute for Vibration Technology, Karlsruhe Technical University; F. Weidenhammer and G. Benz especially gave worthwhile advice. C. W. Mishenkov, from the Moscow Power Institute, assisted me with the preparation of complete references. To all these people, I would like to express my deep thanks.

December 1960   Author - V. V. Bolotin
Moscow

PREFACE TO THE RUSSIAN EDITION

This book is an attempt to present systematically the general theory of dynamic stability of elastic systems and its numerous applications. Investigations of the author are used as the basis for the book, part of which was published previously in the form of separate articles. The author's method of presentation is retained where the problems treated have been analyzed by other authors.

The book is devoted to the solution of technical problems. As in every other engineering (or physics) investigation, the presentation consists of first choosing an initial scheme or pattern, and then using the approximate mathematical methods to obtain readily understood results. This intent, and the desire to make the book easily understood by a large number of readers, is reflected in the arrangement and structure of the book.

The book consists of three parts. PART I is concerned with the simplest problems of dynamic stability which do not require complicated mathematical methods for their solutions. By using these problems, the author wishes to acquaint the reader with previously investigated problems. At the same time, certain peculiarities of the phenomena of instability are clarified, which previously have been only sketchily mentioned. PART I also contains methods of solution of the general problem.

PART II begins with two chapters containing the minimum necessary mathematical information; a conversant reader can disregard these chapters. The properties of the general equations of dynamic stability are then examined; methods are presented for the determination of the boundaries of the regions of instability and the amplitudes of parametrically excited vibrations for the general case.

PART III is concerned with applications. Various problems of the dynamic instability of straight rods, arches, beams, statically indeterminant rod systems, plates, and shells are examined. The choice of examples was dictated by the desire to illustrate the general methods and present solutions to practical problems. The number of examples was limited by the size of the book.

I would like to take this opportunity to express my sincere thanks to A. S. Vol'mir for having read the manuscript and for having given valuable advice.

January 1956 Author - V. V. Bolotin
Moscow

ABSTRACT N.A.

Volume I contains the introduction and the first two chapters of V. V. Bolotin's book, "The Dynamic Stability of Elastic Systems." This work is essentially a systematic exposition of questions in the theory of the dynamic stability of elastic systems. The introduction contains a short history of the development of the subject. Methods for the determination of the boundaries of the regions of dynamic instability are examined ~~in Chapter One~~. The effect of damping on the regions of dynamic instability is investigated. ~~in Chapter Two~~. The results from experimental investigations and an extensive bibliography are also included.

CONTENTS

CONTENTS (continued)

INTRODUCTION

1. In recent years a new branch of the applied theory of elasticity has evolved, the theory of the dynamic stability of elastic systems. In this branch, problems which are examined are related to those in the theory of vibrations and the stability of elastic systems. As in many other areas of learning on the border line of two fields, the theory of dynamic stability is now going through a period of intensive development.

The theory of dynamic stability is easily illustrated by examples.

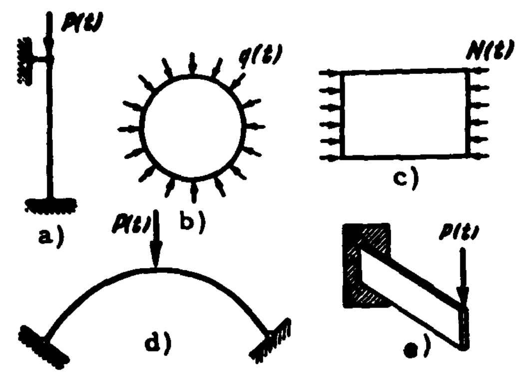

Figure 1

(a) If a straight rod is subjected to a periodic longitudinal load (Figure 1a) and if the amplitude of the load is less than that of the critical static value, generally speaking, the rod experiences only longitudinal vibrations. However, it can be shown that for certain relations between the disturbing frequency θ and the natural frequency of transverse vibrations ω, a straight rod becomes dynamically unstable, and transverse vibrations occur; the amplitude of the transverse vibrations rapidly increases to large values. The relation of the frequencies at which it approaches this resonance (so-called parametric resonance) differs from the frequency relation for the usual resonance of forced oscillations. For sufficiently small values of the amplitude of the longitudinal force,[1] the relation has the form $\theta = 2\omega$.

---

[1] The phenomena of parametric resonance of a stretched string, when one of the ends is attached to an oscillating tuning fork, was uncovered by Melde (1859). The first theoretical explanation of this phenomena was given by Rayleigh (1883-1887). See Ref. 1 for example. A survey of early works on parametric resonance can be found in Zhurn. Tekhn. 4, (1) (1934).

(b) A circular ring compressed by a uniformly distributed radial loading (Figure 1b), in general undergoes only axial deformation. However, for certain relations between the frequency of the load and the natural frequency of the natural bending vibrations of the ring, the initial form of the ring becomes dynamically unstable and develops intense bending vibrations.

(c) Periodic forces acting in the middle plane of a plate (Figure 1c), under certain conditions can excite intense transverse oscillations.

(d) A periodic loading applied symmetrically with respect to the arch (Figure 1d), in general causes only symmetrical vibrations, but under certain conditions can excite asymmetrical vibrations of very large amplitude.

(e) Periodic forces acting on a beam of a narrow cross section and applied in the plane of its greatest rigidity (Figure 1e), under certain conditions can excite bending-torsional vibrations from this plane.

The number of examples can be increased. Whenever static loading of a specific kind makes possible a loss of static stability, vibrational loading of the same kind will make possible a loss of dynamic stability. This is characteristic since loading is contained as a parameter on the left-hand side of the equations of perturbed equilibrium (of motion). We will call such loading parametric; this term is more appropriate, because it indicates the relation to the phenomenon of parametric resonance[2].

By introducing this concept of parametric loading, one can define the theory of the dynamic stability of elastic systems as the study of oscillations originating under action of a pulsating parametric loading[3]. However, it would be more correct to speak not of parametric loadings in general, but

---

[2]In recent years the term has become more generally used by everyone (Ref. 2).

[3]Sometimes the subject of the theory of dynamic stability is interpreted in a broader sense, including problems concerning the vibrations of elastic systems under the effects of certain parametric impact loading. This definition has not been retained.

of loadings parametric with respect to certain forms of the deformations. Thus, a longitudinal force compressing a straight rod is a parametric loading with respect to the transverse deflections, but not with respect to the longitudinal deformations.

2. A detailed review of the literature on the theory of dynamic stability, completed thru 1951, can be found in an article by E. A. Beilin and G. U. Dzhanelidze (Ref. 3). We will look at certain fundamental stages of the development of the theory.

An article by N. M. Beliaev (Ref. 4) published in 1924 can be considered to be the first work on this problem. In this article the problem of dynamic stability of a straight rod hinged on both ends was examined, and the boundaries of the principle region of instability were determined. In 1935, Krylov and Bogliubov (Ref. 5) again returned to the problem and examined the case of general support conditions. Applying the Galerkin variational method, the authors reduced the general problem to the equation which had already been examined by Beliaev, differing only in that the coefficients of the equation are approximate parameters (in the sense of Galerkin's method). A year earlier, Kochin (Ref. 6) examined a mathematically related problem of the vibrations of bent shafts; another related problem was investigated in connection with the oscillations of the driving system of an electric locomotive (Refs. 7 and 8).

We note that the first foreign works on the dynamic stability of rods appear in the late thirties and early forties (Refs. 9, 10, and 11).

The dynamic stability of plates under the action of compressive longitudinal forces was investigated by Bodner (Ref. 12), Khalilov (Ref. 13), Einaudi (Ref. 14), and Ambartsumian and Khachatrian (Ref. 15). The problem of the dynamic stability of a circular ring under the action of a radial pulsating loading was solved by Dzhanelidze and Radtsig (Ref. 16). A number of particular problems were investigated in a pamphlet by Chelomei (Ref. 17). The problem of the dynamic stability of symmetric arches loaded

by compression and bending was investigated by the author (Refs. 18 and 19). Markov (Ref. 20), Oniashvili (Refs. 21 and 22), Bolotin (Ref. 23), Federhofer (Ref. 24*),[4] Yao (Ref. 25*), and Bublik and Merkulov (Ref. 26) investigated certain particular problems on the dynamic stability of shells.

The question of the influence of damping on the boundaries of the regions of instability was discussed by Mettler (Ref. 27) and Naumov (Ref. 28). Let us note that the corresponding problem in a more general form was solved in 1927 by Andonov and Leontovich (Ref. 29). The case of variable loading represented by piece-wise constant segments was investigated by Smirnov (Ref. 30) and Makushin (Ref. 31).

The works mentioned above have a common characteristic in that the problem of dynamic stability is reduced (exactly or approximately) to one differential equation of second order with periodic coefficients (Mathieu-Hill equation). Meanwhile, Chelomei had already shown (Ref. 17) that the problem of dynamic stability in the general case reduces to systems of differential equations with periodic coefficients. Brachkovski (Ref. 32) (using the method of Galerkin) and the author (Ref. 33) (using integral equations) established a class of problems which can be exactly reduced to one equation of second order. A generalization of these results for the case of dissipative systems was given by Dzhanelidze (Ref. 34). Although the properties of the equations obtained by the Galerkin method have been well studied, the number of publications based on this approximation continues to grow (Ref. 35).

In certain papers (Refs. 36, 37, and 38), the problem of the stability of plane bending, requiring an examination of a system of differential equations, is reduced to one Mathieu-Hill equation.

I. I. Gol'denblat (Ref. 39) investigated the problem of the stability of a compressed thin-walled rod, symmetrical about one axis. The problem was reduced to a system of two differential equations. Using the results of

[4] References followed by an asterisk have been added by the translator.

N. A. Artem'ev (Ref. 40), Gol'denblat presented a method of constructing the regions of instability by means of expanding according to the power of the small parameter. A similar method, devoid however of a rigorous foundation, was applied by Mettler (Ref. 41) to the problem of dynamic stability of the plane bending of a beam. Weidenhammer (Ref. 42), using the same method, investigated the problem of the stability of a rod clamped on the ends. Another version of the method is given by Kucharski (Ref. 43) in the application to the special problem of the dynamic stability of plates and by Reckling (Refs. 44, 45, and 46) in the application to the special problem of the dynamic stability of plane bending. Still another variation of this method was applied by Yakubovich (Refs. 47, 48, and 49).

Another method, free from the assumption of the small parameter, is given in an article by the author (Ref. 33). This article also investigated the structure of the general equations of dynamic stability. The author (Ref. 19) and Piszczek (Refs. 50, 51, 52, and 53) also investigated the general problem of the dynamic stability of plane bending; the author (Ref. 54) investigated the problem of the dynamic stability of plates. In the present book this method is extended to dissipation systems and is applied systematically to problems involving the stability of rods, arches, beams, frames, plates and shells.

In the works enumerated above, the problem of dynamic stability was examined in the sense of finding the regions, in the boundaries of which a given form of the motion becomes dynamically unstable. The idea of the inadequacy of the linear treatment for determining accurate values of the amplitudes in the resonance regions was first clearly formulated by Gol'denblat (Ref. 55), who indicated a relation of this problem with those involving the excitation of electrical oscillations (see Ref. 56). The presentation of a nonlinear theory applicable to the problem of the dynamic stability of a compressed rod was given by the author (Ref. 57). An analogous problem was examined almost simultaneously by Weidenhammer (Ref. 58); see also Refs. 59 and 60). In a paper by the author (Ref. 61) the nonlinear theory is extended to the secondary regions of instability and also to the case of a rod

having initial curvature. Other nonlinear problems were investigated by Bolotin (Refs. 62 and 63) and Ivovich (Refs. 64, 65, 66, and 67). In another article (Ref. 68), the solution of the related problem of the oscillations of a rotating shaft having different principal bending stiffnesses is given. Certain nonlinear problems on the dynamic stability of plates and shells were investigated by the author (Refs. 69 and 23) which included the stretching of the middle surface. The present book gives the solution of a range of new nonlinear problems, and in particular, for arches, beams, and statically indeterminate rod systems.

Presently very little experimental data exist on this subject, although these data represent a field of definite interest. Experiments of parametric excited transverse vibrations of a compressed rod are described by the author (Ref. 57). These experiments determine the amplitudes of steady-state vibrations and investigate damping, the regime of beating, and the process of establishing the vibrations . This paper also gives a comparison of the experimental results with theoretical results. The parametric excited vibrations of compressed-curved arches are described in previous papers of the author (Refs. 18 and 70). Experiments on the dynamic stability of plane bending of beams were conducted by I. A. Burnashev (Ref. 71) and V. A. Sobolev (Ref. 72).

3. The theory of dynamic stability has already opened the way for direct engineering applications. Parametrically excited vibrations are similar in appearance, on the surface, with the accompanying forced oscillations and can therefore qualify as ordinary resonance vibrations by practical engineering standards. In a number of cases, however, the usual procedures of damping and vibration insulation may break down in the case of periodic vibrations and even bring the opposite results. Although the vibrations may not threaten the structure or its normal operation, they can bring about fatigue failure if they continue to act. Therefore, the study of the formation of parametric vibrations and methods of the prevention of their occurrence is necessary for engineering researchers in the various areas of mechanics, transportation, and industrial structures.

The theory of dynamic stability is one of the newest branches of the mechanics of deformable solids. Although during the last ten years much has already been done to clarify many problems that were only recently completely obscure, a large and beneficial field for investigations remains.

PART I
Elementary Problems of Dynamic Stability

CHAPTER ONE

DETERMINATION OF THE REGIONS OF DYNAMIC INSTABILITY

•1. DIFFERENTIAL EQUATION OF THE PROBLEM

1. Consider the problem of the transverse oscillations of a straight rod loaded by a periodic longitudinal force (Fig. 2). The rod is assumed to be simply supported and of uniform cross section along its length. We shall make the usual assumptions of strength of materials, i. e., that Hooke's law holds and plane sections remain plane. The case of nonlinear elasticity will be examined in Chapter III and further on in the book.

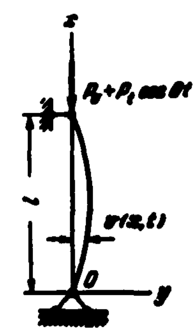

Figure 2

This problem is similar to a number of the simplest problems of dynamic stability. It was first set down precisely in such a form by N. M. Beliaev (Ref. 4).

We will proceed from the well-known equation of static bending

$$EJ\frac{d^2v}{dx^2}+Pv=0,$$

where v(x) is the deflection of the rod, EJ is its rigidity during bending, and P is the longitudinal force. After two differentiations, the equation takes the form

$$EJ\frac{d^4v}{dx^4}+P\frac{d^2v}{dx^2}=0; \tag{1.1}$$

which gives the condition that the sum of the y components of all the forces per unit length acting on the rod is equal to zero.

To arrive at the equation for the transverse oscillations of a rod under the action of the periodic longitudinal force,

$$P(t)=P_0+P_t\cos\theta t,$$

it is necessary to introduce additional terms into Eq. (1. 1) which take into account the inertia forces (Ref. 9).

As in the case of the applied theory of vibrations, we will not include the inertia forces associated with the rotation of the cross sections of the rod with respect to its own principal axes. The influence of longitudinal inertia forces will be considered in later chapters. In the meantime, note that longitudinal inertia forces can substantially influence the dynamic stability of a rod only in the case when the frequency of the external force is close to the longitudinal natural frequencies of the rod, i. e., when the longitudinal vibrations have a resonance character. In the following discussion, we will consider that the system is not close to the resonance of the longitudinal vibrations.

With these reservations, the inertia forces acting on the rod can be reduced to a distributed loading, the magnitude of which is

$$-m\frac{\partial^2 v}{\partial t^2},$$

where m is the mass of the rod per unit length. Thus, we arrive at the following equation,

$$EJ\frac{\partial^4 v}{\partial x^4}+(P_0+P_t\cos\theta t)\frac{\partial^2 v}{\partial x^2}+m\frac{\partial^2 v}{\partial t^2}=0, \tag{1.2}$$

for the dynamic deflections v(x, t) of the rod at any arbitrary instant of time.

2. We will seek the solution of Eq. (1.2) in the form

$$v(x,\ t)=f_k(t)\sin\frac{k\pi x}{l} \qquad (k=1,\ 2,\ 3,\ \ldots), \tag{1.3}$$

where $f_k(t)$ are unknown functions of time and l is the length of the rod. One easily sees that Eq. (1.3) satisfies the boundary conditions of the problem, requiring in the given case that the deflection together with its second derivatives vanish at the ends of the rod. We shall remind the reader that the "fundamental functions"

$$\varphi_k(x)=\sin\frac{k\pi x}{l}$$

are of the same form as that of the natural vibrations and the loss of static stability of a freely supported rod.

Substitution of Eq. (1. 3) into Eq. (1. 2) gives

$$\left[m\frac{d^2f_k}{dt^2}+EJ\frac{k^4\pi^4f_k}{l^4}-(P_0+P_t\cos\theta t)\frac{k^2\pi^2f_k}{l^2}\right]\sin\frac{k\pi x}{l}=0.$$

For Eq. (1. 3) to really satisfy Eq. (1. 2), it is necessary and sufficient that, at any t, the quantity in the square bracket should vanish. In other words, the functions $f_k(t)$ must satisfy the differential equation

$$\frac{d^2f_k}{dt^2}+\omega_k^2\left(1-\frac{P_0+P_t\cos\theta t}{P_{*k}}\right)f_k=0 \quad (k=1,\ 2,\ 3,\ \ldots) \tag{1.4}$$

The notation

$$\omega_k=\frac{k^2\pi^2}{l^2}\sqrt{\frac{EJ}{m}} \tag{1.5}$$

is introduced into Eq. (1. 4) for the kth frequency of the free vibrations of an unloaded rod and,

$$P_{*k}=\frac{k^2\pi^2EJ}{l^2}. \tag{1.6}$$

for the kth Euler buckling force (the asterisk denotes this given quantity in future problems).

It is convenient to represent Eq. (1.4) in the form

$$\frac{d^2 f_k}{dt^2} + \Omega_k^2 (1 - 2\mu_k \cos \theta t) f_k = 0 \qquad (k = 1, 2, 3, \ldots), \tag{1.7}$$

where Ω_k is the frequency of the natural oscillations of the rod loaded by a constant longitudinal force P_o,

$$\Omega_k = \omega_k \sqrt{1 - \frac{P_0}{P_{*k}}}, \tag{1.8}$$

and μ_k is a quantity which we will call the <u>coefficient of excitation</u>

$$\mu_k = \frac{P_t}{2(P_{*k} - P_0)}. \tag{1.9}$$

Because Eq. (1.7) is identical for all forms of oscillations, i.e., for all k, we will in the future omit the indices of Ω_k and μ_k and write this equation in the form

$$f'' + \Omega^2 (1 - 2\mu \cos \theta t) f = 0. \tag{1.10}$$

The prime denotes differentiation with respect to time.

3. Equation (1.10) is the well known Mathieu equation. In the more general case of the longitudinal force given by

$$P(t) = P_0 + P_t \Phi(t),$$

where Φ (t) is a periodic function with a period T

$$\Phi(t+T) = \Phi(t),$$

we arrive at

$$f'' + \Omega^2 [1 - 2\mu \Phi(t)] f = 0.$$

Such an equation, more general than the Mathieu equation, is usually called Hill's equation.

Mathieu-Hill equations are encountered in different areas of physics and engineering. Certain problems in theoretical physics are reduced to a similar equation, in particular the problem of the propagation of electromagnetic waves in a medium with a periodic structure. In the quantum theory of metals, the problem of the motion of electrons in a crystal lattice reduces to the Mathieu-Hill equation. The Mathieu-Hill equation is also encountered in the investigations of the stability of the oscillatory processes in nonlinear systems, in the theory of the parametric excitation of electrical oscillations, and other divisions of the theory of oscillations. Certain problems of celestial mechanics and cosmogony, in particular the theory of the motion of the moon, also lead to Hill's equation.

A vast amount of literature is devoted to the investigation of the Mathieu-Hill equation (see, for example, Refs. 73, 74, and 75). One of the most interesting characteristics of this equation is that for certain relations between its coefficients, it has solutions which are unbounded. The values of the coefficients cover certain regions in the plane of the two parameters μ and Ω, i.e., the regions which in the physical problem under consideration correspond to the regions of dynamic instability.

For example, Fig. 3 shows the distribution of the regions of instability for the Mathieu equation

$$\frac{d^2f}{dx^2}+(\lambda-h^2\cos 2x)f=0.$$

In such a form, the coefficients of the equation depend on the two parameters λ and h^2 which are plotted as coordinates. The regions in which the solutions of the equation are unbounded are crosshatched. As is evident from the figure, the regions of instability occupy a considerable part of the plane of the parameters.

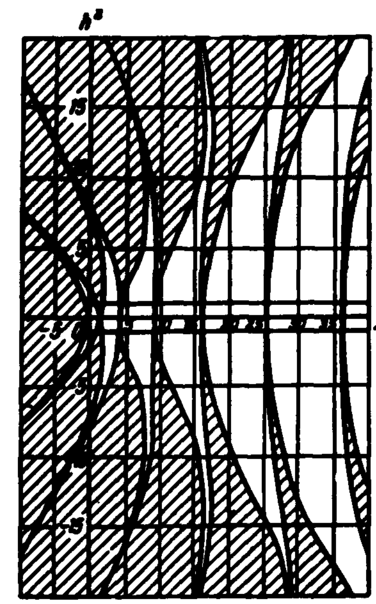

Figure 3

Therefore, to answer the question concerning the dynamic stability of a rod, it is necessary to find on the λ, h^2 plane a point corresponding to the given ratio of parameters. If a point occurs in the non-crosshatched region, it means that the initial straight form of the rod is dynamically stable. If, however, the same point is found in the crosshatched region, then any initial deviation of the straight form of the rod will increase indefinitely with time, i.e., the straight form of the rod will be dynamically unstable.

The determination of the regions of dynamic instability constitutes one of the main tasks of the theory.

•2. SOME PROPERTIES OF THE MATHIEU-HILL EQUATION

1. Consider the equation

$$f'' + \Omega^2 [1 - 2\mu\Phi(t)] f = 0, \tag{1.11}$$

in which $\Phi(t)$ is a period function with a period of

$$T = \frac{2\pi}{\theta}. \tag{1.12}$$

With respect to this function, we shall assume that it can be represented in the form of the converging Fourier series

$$\Phi(t) = \sum_{k=1}^{\infty} (\mu_k \cos k\theta t + \nu_k \sin k\theta t). \tag{1.13}$$

Note, first of all, that Eq. (1.11) does not change its form by adding the period to t. This follows from the fact that

$$\Phi(t+T) = \Phi(t).$$

Therefore, if f(t) is a solution of Eq. (1.11), then f(t+T) is also its solution.

Let f_1 (t) and f_2 (t) be any two linearly independent solutions of Eq. (1.11). Then on the basis of the previous discussion, f_1 (t+T) and f_2(t+T) are also its solution, and, consequently can be represented in the form of a linear combination of the primary functions

$$\left.\begin{aligned} f_1(t+T) &= a_{11}f_1(t) + a_{12}f_2(t), \\ f_2(t+T) &= a_{21}f_1(t) + a_{22}f_2(t). \end{aligned}\right\} \tag{1.14}$$

Here a_{ik} are certain constants.

Thus, the addition of the period to t results in a linear transformation of the initial system of solutions. If, instead of initially chosen solutions $f_{1,2}$(t), we take some other linearly independent solution, then the coefficients of the transformation of Eq. (1.14) generally will change. In particular, one can try to choose solutions $f^*_{1,2}$(t) such that the secondary coefficients in Eq. (1.14) vanish

$$a_{12} = a_{21} = 0.$$

The transformation of the characteristics in this case will take its simpliest form. It is reduced to the simple multiplication of functions by certain constants[1]

$$\left.\begin{aligned} f_1^*(t+T) &= \rho_1 f_1^*(t), \\ f_2^*(t+T) &= \rho_2 f_2^*(t). \end{aligned}\right\} \tag{1.15}$$

In contrast to Eq. (1.14), we will introduce here the new notations

$$a_{11} = \rho_1,$$
$$a_{22} = \rho_2.$$

[1]These solutions are the well-known Floquet Solutions.

It is known from the theory of linear transformations (for example, see Chapter X) that any transformation of the type in Eq. (1.14) can be reduced to the simplest or, as is more commonly referred to, the diagonal form where the numbers $\rho_{1,2}$ are determined from the characteristic equation[2]

$$\begin{vmatrix} a_{11}-\rho & a_{12} \\ a_{21} & a_{22}-\rho \end{vmatrix} = 0, \qquad (1.16)$$

2. The characteristic equation plays an important role in the theory of the Mathieu-Hill equation since, as we will see below, it defines the character of solution of Mathieu-Hill equation in many respects. We will now show what this equation consists of.

Let $f_1(t)$ and $f_2(t)$ be two linearly independent solutions of Eq. (1.11) satisfying the initial conditions

$$\left.\begin{array}{ll} f_1(0)=1, & f_1'(0)=0. \\ f_2(0)=0, & f_2'(0)=1. \end{array}\right\} \qquad (1.17)$$

Then, letting $t = 0$ in Eq. (1.14), we obtain

$$a_{11}=f_1(T),$$
$$a_{21}=f_2(T).$$

Differentiating Eq. (1.14) termwise and letting $t = 0$, we have

$$a_{12}=f_1'(T),$$
$$a_{22}=f_2'(T).$$

[2] For the sake of simplicity, we have omitted here one detail which will be explained later — the case where the characteristic equation has multiple roots of nonlinear elementary divisors.

Thus, the characteristic equation takes the form

$$\begin{vmatrix} f_1(T)-\rho & f_1'(T) \\ f_2(T) & f_2'(T)-\rho \end{vmatrix} = 0$$

or, if one expands the determinant, it takes the form

$$\rho^2 - 2A\rho + B = 0. \tag{1.18}$$

In Eq. (1.18) the following notations are assumed

$$A = \frac{1}{2}[f_1(T) + f_2'(T)]$$
$$B = f_1(T)f_2'(T) - f_2(T)f_1'(T).$$

By their very meaning, the roots of the characteristic equation, and consequently its coefficients do not depend on the choice of the solutions $f_{1,2}(t)$. One can show, for example, term B of the characteristic equation is always equal to unity. Because the functions $f_{1,2}(t)$ are solutions of Eq. (1.11), then

$$f_1'' + \Omega^2[1 - 2\mu\Phi(t)]f_1 = 0,$$
$$f_2'' + \Omega^2[1 - 2\mu\Phi(t)]f_2 = 0.$$

Multiplying the first of these identities by $f_2(t)$, the second by $f_1(t)$, and subtracting one from the other, we obtain

$$f_1(t)f_2''(t) - f_2(t)f_1''(t) = 0,$$

after integrating we obtain

$$f_1(t)f_2'(t)-f_2(t)f_1'(t)=\text{const.}$$

The quantity on the left-hand side coincides, for t = T, with the B term in Eq. (1.18). For the determination of the constant on the right-hand side, we we shall set t = 0. Then, making use of the initial conditions in Eq. (1.17), we will find

$$f_1(T)f_2'(T)-f_2(T)f_1'(T)=1,$$

Thus, the characteristic equation takes the form

$$\rho^2-2A\rho+1=0; \tag{1.19}$$

its roots, obviously, are connected by the relationship

$$\rho_1\cdot\rho_2=1. \tag{1.20}$$

3. It was shown in No. 1 that among the particular solutions of Eq. (1.1) there exist two linearly independent solutions $f^*_{1,2}(t)$ satisfying Eq. (1.15)

$$f_k^*(t+T)=\rho_k f_k^*(t) \qquad (k=1,\ 2).$$

These solutions, which acquire a constant multiplier by the addition of the period to t, can be represented in the form

$$f_k^*(t)=\chi_k(t)e^{\frac{t}{T}\ln\rho_k} \qquad (k=1,\ 2). \tag{1.21}$$

where $\chi_{1,2}(t)$ are certain periodic functions of period T. In fact,

$$f_k(t+T) = \chi_k(t)\, e^{\left(\frac{t}{T}+1\right)\ln \rho_k} = \rho_k f_k(t).$$

It follows from Eq. (1.21) that the behavior of the solutions as $t \to \infty$ depends on the value of the characteristic roots (more precisely, on the value of its moduli). In fact, taking into account that

$$\ln \rho = \ln |\rho| + i \arg \rho,$$

we can rewrite Eq. (1.21) in the following form

$$f_k(t) = \varphi_k(t)\, e^{\frac{t}{T}\ln|\rho_k|} \qquad (k=1,\ 2), \tag{1.22}$$

where $\phi_k(t)$ is the bounded (almost periodic) function

$$\varphi_k(t) = \chi_k(t)\, e^{\frac{it}{T}\arg\rho}.$$

If the characteristic number ρ_k is greater than unity, then the corresponding solution, Eq. (1.22), will have an unbounded exponential multiplier. If the same characteristic number is less than unity, then the corresponding solution is damped with increasing t. Finally, if the characteristic number is equal to unity, then the solution has a periodic (or almost periodic) character, i.e., it will be bounded in time.

Let

$$|A| = \frac{1}{2}|f_1(T) + f_2'(T)| > 1.$$

Then, as can be seen from Eq. (1. 19), the characteristic roots will be real, and one of them will be greater than unity. In this case the general integral of Eq. (1. 11) will unboundedly increase with time

$$f(t)=C_1\chi_1(t)e^{\frac{t}{T}\ln\rho_1}+C_2\chi_2(t)e^{\frac{t}{T}\ln\rho_2}.$$

However, if

$$\frac{1}{2}|f_1(T)+f_2'(T)|<1,$$

the characteristic equation has conjugate complex roots, and since their product must be equal to unity, their modulus will be equal to unity. The case of complex characteristic roots corresponds to the region of bounded solutions. On the boundaries separating the regions of the bounded solutions from the regions where the general integral unboundedly increases with time, the following condition must be satisfied.

$$|f_1(T)+f_2'(T)|=2. \tag{1.23}$$

One can make use of the Eq. (1. 23) to determine the boundaries of dynamic instability. However for its construction, it is necessary to know particular solutions of the problem, at least during the first period of oscillation. This calculation, however, has serious computational difficulties connected with it. Only in certain special cases is it possible to integrate the differential equation of the type of Eq. (1. 11) in terms of elementary functions. One of these cases will be considered in the following paragraph.

•3. CONSTRUCTION OF THE REGIONS OF DYNAMIC INSTABILITY FOR A PARTICULAR CASE

Let the longitudinal force change according to the piecewise-constant law, i.e., during the first period

$$P(t) = P_0 + P_t, \quad \text{if} \quad 0 < t < t_0,$$
$$P(t) = P_0 - P_t, \quad \text{if} \quad t_0 < t < T.$$

Such a law of changing load is rarely met in practice. However, in the case when $t_0 = (T/2)$, we have a variation which for a small P_t can be considered as a first crude approximation to the harmonic regime

$$P(t) = P_0 + P_t \sin \theta t.$$

This is the case to which we shall restrict ourselves in future considerations.

It is possible to write down the equation of the vibrations in the form

$$f'' + \Omega^2 [1 - 2\mu\Phi(t)] f = 0,$$

where

$$\Phi(t) = 1, \quad \text{if} \quad 0 < t < \frac{T}{2},$$
$$\Phi(t) = -1, \quad \text{if} \quad \frac{T}{2} < t < T,$$

and the coefficient of excitation, as before, is equal to

$$\mu = \frac{P_t}{2(P_* - P_0)}.$$

During the first half of the period, the vibrations are represented by a differential equation with constant coefficients

$$f'' + \Omega^2(1 - 2\mu)f = 0.$$

Its general solution, as is well-known, will be

$$f(t) = C_1 \sin p_1 t + D_1 \cos p_1 t,$$

where, for the sake of brevity

$$p_1 = \Omega\sqrt{1 - 2\mu}.$$

The particular solution, satisfying the initial conditions $f_1(0) = 1$, $f_1'(0) = 0$, is

$$f_1(t) = \cos p_1 t.$$

The second solution, satisfying the initial conditions $f_2(0) = 0$, $f_2'(0) = 1$, obviously will be

$$f_2(t) = \frac{1}{p_1} \sin p_1 t.$$

These two solutions must be extended to the second interval of time $(T/2) < t \leq T$, during which the vibrations are described by

$$f'' + \Omega^2(1 + 2\mu)f = 0.$$

The general solution of this equation will be

$$f(t) = C_2 \sin p_2 t + D_2 \cos p_2 t,$$

where $p_2 = \Omega\sqrt{1 + 2\mu}$, similar to the previous case. The constants C_2 and D_2 must be found from the condition that on the boundary of the two half-periods

(at $t = T/2$), the functions $f_{1,2}(t)$ and their first derivatives be continuous

$$f_{1,2}\left(\frac{T}{2}-\varepsilon\right)=f_{1,2}\left(\frac{T}{2}+\varepsilon\right), \qquad (\varepsilon \to 0).$$
$$f'_{1,2}\left(\frac{T}{2}-\varepsilon\right)=f'_{1,2}\left(\frac{T}{2}+\varepsilon\right)$$

Substitution gives, for the continuation of the function $f_1(t)$,

$$\cos\frac{p_1T}{2}=C_2\sin\frac{p_2T}{2}+D_2\cos\frac{p_2T}{2},$$
$$-p_1\sin\frac{p_1T}{2}=p_2C_2\cos\frac{p_2T}{2}-p_2D_2\sin\frac{p_2T}{2}.$$

Solving these equations with respect to the constants C_2 and D_2, and replacing $T = (2\pi)/\theta$, we find

$$C_2=\cos\frac{\pi p_1}{\theta}\sin\frac{\pi p_2}{\theta}-\frac{p_1}{p_2}\sin\frac{\pi p_1}{\theta}\cos\frac{\pi p_2}{\theta},$$
$$D_2=\cos\frac{\pi p_1}{\theta}\cos\frac{\pi p_2}{\theta}+\frac{p_1}{p_2}\sin\frac{\pi p_1}{\theta}\sin\frac{\pi p_2}{\theta}.$$

Similarly, for the function $f_2(t)$

$$C_2=\frac{1}{p_1}\sin\frac{\pi p_1}{\theta}\sin\frac{\pi p_2}{\theta}+\frac{1}{p_2}\cos\frac{\pi p_1}{\theta}\cos\frac{\pi p_2}{\theta},$$
$$D_2=\frac{1}{p_1}\sin\frac{\pi p_1}{\theta}\cos\frac{\pi p_2}{\theta}-\frac{1}{p_2}\cos\frac{\pi p_1}{\theta}\sin\frac{\pi p_2}{\theta}.$$

Substituting the values of coefficients C_2 and D_2 in the expression for $f(t)$, we can calculate

$$A=\frac{1}{2}[f_1(T)+f_2'(T)].$$

After a number of cumbersome transformations, we obtain

$$A = \cos\frac{\pi p_1}{\theta}\cos\frac{\pi p_2}{\theta} - \frac{p_1^2 + p_2^2}{2p_1p_2}\sin\frac{\pi p_1}{\theta}\sin\frac{\pi p_2}{\theta}. \tag{1.24}$$

In conformity with the results of the preceding paragraph, we can conclude that for

$$\left|\cos\frac{\pi p_1}{\theta}\cos\frac{\pi p_2}{\theta} - \frac{p_1^2 + p_2^2}{2p_1p_2}\sin\frac{\pi p_1}{\theta}\sin\frac{\pi p_2}{\theta}\right| < 1$$

the equation of the problem under consideration does not have unbounded increasing solutions—the initial straight form of the rod is dynamically stable. For

$$\left|\cos\frac{\pi p_1}{\theta}\cos\frac{\pi p_2}{\theta} - \frac{p_1^2 + p_2^2}{2p_1p_2}\sin\frac{\pi p_1}{\theta}\sin\frac{\pi p_2}{\theta}\right| > 1,$$

the amplitudes of the transverse oscillations will unboundedly increase with time. The equation

$$\left|\cos\frac{\pi p_1}{\theta}\cos\frac{\pi p_2}{\theta} - \frac{p_1^2 + p_2^2}{2p_1p_2}\sin\frac{\pi p_1}{\theta}\sin\frac{\pi p_2}{\theta}\right| = 1 \tag{1.25}$$

permits one to determine the boundaries of the regions of dynamic instability. This equation can be found in many references. (See, for example, Ref. 76)

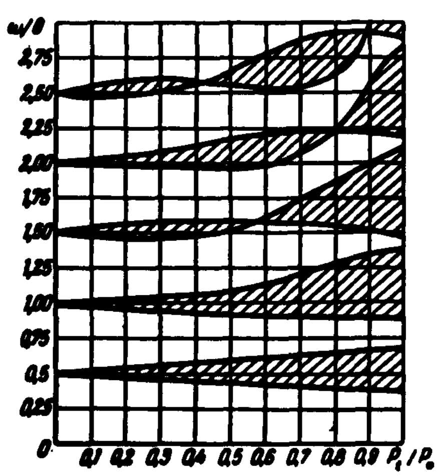

Figure 4

Detailed calculations are carried out in the work by V. M. Makushin (Ref. 30). One of the diagrams from this work is shown in Fig. 4. The regions of instability are cross-hatched.

Equation (1.25) can be generalized for the case when the longitudinal force changes

according to an arbitrary piecewise-constant law. The corresponding equation has the form

$$\left|\left(2\cos p_1t_0 \sin p_2t_0 - \frac{p_1^2+p_2^2}{p_1p_2}\sin p_1t_0 \cos p_2t_0\right)\sin p_2T + \left(2\cos p_1t_0 \cos p_2t_0 + \frac{p_1^2+p_2^2}{p_1p_2}\sin p_1t_0 \sin p_2t_0\right)\cos p_2T\right| = 1.$$

A discussion of the results will be postponed until later.

•4. DERIVATION OF THE CRITICAL FREQUENCY EQUATION

1. A method of determining the regions of instability is presented in the following for the case of an arbitrary periodic function, given in series form in Eq. (1.13).

It was shown in •2 that the region of real characteristic numbers coincides with the region which has unboundedly increasing solutions of Eq. (1.11). On the other hand, the region of complex characteristic roots corresponds to the bounded (almost periodic) solutions.

Multiple roots occur on the boundaries dividing the regions of real and complex roots; moreover, as follows from Eq. (1.20), such roots can be either $\rho_1 = \rho_2 = 1$, or $\rho_1 = \rho_2 = -1$.

In the first case, as seen from Eq. (1.15), the solution of the differential equation will be periodic with a period $T = (2\pi)/\theta$; in the second case[3] we will have the period 2T.

Therefore, the regions with unboundedly increasing solutions are separated from the regions of stability by the periodic solutions with a period

[3]A more detailed analysis shows that only one of the particular solutions will be periodic. The second solution will have the form

$$f(t) = \chi_1(t) + t\chi_2(t)$$

where $\chi_1(t)$ and $\chi_2(t)$ are periodic functions of time.

T and 2T. More exactly, two solutions of identical periods bound the region of instability, two solutions of different periods bound the region of stability.

The last property is obtained easily from the following considerations. Assume that in the interval between $\rho = 1$ and $\rho = -1$ lies the region of real roots (the region of instability). As a consequence of the continuous dependence of the characteristic roots on the coefficients of the differential equation, there must then be among them the root $\rho = 0$, and consequently, also $\rho = \infty$, which is impossible. Thus, the roots $\rho = 1$ and $\rho = -1$ bound the region of the complex roots, i.e., the region of stability.[4]

2. From the preceding discussion, it follows that the determination of the boundaries of the regions of instability is reduced to finding the conditions under which the given differential equation has periodic solutions with periods T and 2T. From the viewpoint of the physical problem considered here, such results seem completely natural. Indeed the periodic motion is essentially the boundary case for oscillations with unboundedly increasing amplitudes.

To find conditions for the existence of periodic solutions, we can often proceed in the following manner (see, for example, Ref. 7). Having introduced the "small parameter" μ (the coefficient of excitation, for example, can be accepted as such a parameter), one seeks the solution to

$$f'' + \Omega^2 [1 - 2\mu\Phi(t)] f = 0$$

in the form of a power series of μ

$$f = f_0 + \mu f_1 + \mu^2 f_2 + \cdots$$

Here f_k are unknown functions of time. Substituting this expression in the initial equation and equating the coefficients having the same μ^k, one obtains

[4]A rigorous proof of this theorem can be found in the book by M. J. O. Strutt, Ref. 73.

a system of differential equations with constant coefficients which can be solved by the method of successive approximations. The solutions found in this manner have limitations imposed on them in the form of the requirement of absence of infinite terms, i. e., the requirement of the periodicity of solutions.

However, the conditions for the existence of periodic solutions can be obtained in a different manner, i. e., without the application of the method of perturbation of small parameters borrowed from non-linear mechanics. The fact that the periodic solutions do exist and that they can be expanded into Fourier series is known. This permits one to seek the periodic solutions of Eq. (1. 11) directly in trigometric series form. As an example, we shall apply this method to the Mathieu equation

$$f'' + \Omega^2(1 - 2\mu\cos\theta t)f = 0. \tag{1.26}$$

We seek the periodic solution with a period 2T in the form

$$f(t) = \sum_{k=1,3,5}^{\infty}\left(a_k\sin\frac{k\theta t}{2} + b_k\cos\frac{k\theta t}{2}\right). \tag{1.27}$$

Substituting Eq. (1. 27) into Eq. (1. 26), and equating the coefficients of identical sin $(k\theta t)/2$ and cos $(k\theta t)/2$ gives the following system of linear homogeneous algebraic equations in terms of a_k and b_k;

$$\begin{gathered}
\left(1+\mu-\frac{\theta^2}{4\Omega^2}\right)a_1-\mu a_3=0,\\
\left(1-\frac{k^2\theta^2}{4\Omega^2}\right)a_k-\mu(a_{k-2}+a_{k+2})=0 \quad (k=3,\ 5,\ 7,\ldots),\\
\left(1-\mu-\frac{\theta^2}{4\Omega^2}\right)b_1-\mu b_3=0,\\
\left(1-\frac{k^2\theta^2}{4\Omega^2}\right)b_k-\mu(b_{k-2}+b_{k+2})=0 \quad (k=3,\ 5,\ 7,\ldots).
\end{gathered}$$

Note that the first system contains only coefficients a_k, the second contains only b_k.

3. As is well known, the system of linear homogeneous equations has solutions different from zero only in the case when the determinant composed of the coefficients of this system is equal to zero. This also holds in the case when the system contains an infinite number of unknowns. Thus, the fact that the determinants of the homogeneous systems obtained are equal to zero is the condition for the existence of the periodic solution of Eq. (1.26). Joining the two conditions under the ± sign, we obtain

$$\begin{vmatrix} 1 \pm \mu - \frac{\theta^2}{4\Omega^2} & -\mu & 0 & \cdots \\ -\mu & 1 - \frac{9\theta^2}{4\Omega^2} & -\mu & \cdots \\ 0 & -\mu & 1 - \frac{25\theta^2}{4\Omega^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \tag{1.28}$$

This equation relating the frequencies of the external loading with the characteristic frequency of the rod and the magnitude of the external force will be called <u>the equation of critical frequencies</u>, where <u>critical</u> frequencies are understood to be the frequencies of the external loading θ corresponding to the boundaries of the regions of instability. Equation (1.28) makes it possible to find regions of instability which are bounded by the periodic solutions with a period 2T. To determine the regions of instability bounded by the periodic solutions with a period T, we proceed in an analogous manner. Having substituted into Eq. (1.26) the series.

$$f(t) = b_0 + \sum_{k=2,4,6}^{\infty} \left(a_k \sin \frac{k\theta t}{2} + b_k \cos \frac{k\theta t}{2} \right),$$

we obtain the following systems of algebraic equations:

$$\left(1 - \frac{\theta^2}{\Omega^2}\right) a_2 - \mu a_4 = 0,$$

$$\left(1 - \frac{k^2\theta^2}{4\Omega^2}\right) a_k - \mu (a_{k-2} + a_{k+2}) = 0, \qquad (k = 4,\ 6,\ \ldots),$$

$$b_0 - \mu b_2 = 0,$$

$$\left(1 - \frac{\theta^2}{\Omega^2}\right) b_2 - \mu (2b_0 + b_4) = 0,$$

$$\left(1 - \frac{k^2\theta^2}{4\Omega^2}\right) b_k - \mu (b_{k-2} + b_{k+2}) = 0 \qquad (k = 4,\ 6,\ \ldots).$$

Equating the determinant of the obtained homogeneous system to zero, we arrive at the following equations for the critical frequencies

$$\begin{vmatrix} 1-\frac{\theta^2}{\Omega^2} & -\mu & 0 & \cdots \\ -\mu & 1-\frac{4\theta^2}{\Omega^2} & -\mu & \cdots \\ 0 & -\mu & 1-\frac{16\theta^2}{\Omega^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \qquad (1.29)$$

and

$$\begin{vmatrix} 1 & -\mu & 0 & 0 & \cdots \\ -2\mu & 1-\frac{\theta^2}{\Omega^2} & -\mu & 0 & \cdots \\ 0 & -\mu & 1-\frac{4\theta^2}{\Omega^2} & -\mu & \cdots \\ 0 & 0 & -\mu & 1-\frac{16\theta^2}{\Omega^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \qquad (1.30)$$

4. The determinants obtained in No. 3 are infinite and, therefore, the question of their convergence must be considered.[5]

One can show that these determinants belong to a well-known class of converging determinants, i.e., to normal determinants. The determinant

$$\Delta = \begin{vmatrix} 1+c_{11} & c_{12} & c_{13} & \cdots \\ c_{21} & 1+c_{22} & c_{23} & \cdots \\ c_{31} & c_{32} & 1+c_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (1.31)$$

[5]See Ref. 77. Infinite determinants were investigated for the first time in connection with the integration of the Hill equation (Lunar Theory, 1877).

is called normal if the double series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} c_{ik}.$$

is absolutely convergent.

We can examine, for example, the determinant in Eq. (1.28). When we multiply the kth row (k = 1, 2, 3, . . .) by $-4\Omega^2/[2k-1)^2\theta^2]$, it can be reduced to the same form as Eq. (1.3.), where

$$c_{kk} = \begin{cases} -\frac{4\Omega^2}{\theta^2}(1 \pm \mu) & (k = 1), \\ -\frac{4\Omega^2}{(2k-1)^2\theta^2} & (k \neq 1); \end{cases}$$

$$\underset{(i \neq k)}{c_{ik}} = \begin{cases} -\frac{4\Omega^2}{(2k-1)^2\theta^2}\mu & (i = k \pm 1), \\ 0 & (i \neq k \pm 1). \end{cases}$$

Constructing the double series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} c_{ik},$$

we can prove that it converges absolutely. Actually we have the inequality

$$\sum_{i=1}^{n} \sum_{k=1}^{n} |c_{ik}| < \frac{4\Omega^2}{\theta^2}(1 + 2\mu) \sum_{k=1}^{n} \frac{1}{(2k-1)^2},$$

where the series standing on the right side is convergent.

One can prove analogously the convergence of the remaining determinants.

•5. DETERMINATION OF THE REGIONS OF DYNAMIC INSTABILITY

1. For the clarification of the general character of the distribution of the regions of instability, we will examine the case of a very small periodic component of the longitudinal force. Letting $\mu \to 0$ in Eqs. (1.28), (1.2)

and (1.30), we find that for very small values of μ the solutions with a period 2T lie in pairs in the vicinity of the frequencies

$$\theta_* = \frac{2\Omega}{k} \quad (k = 1, 3, 5, \ldots),$$

and with a period T in the vicinity of the frequencies

$$\theta_* = \frac{2\Omega}{k} \quad (k = 2, 4, 6, \ldots).$$

Both cases can be combined in one formula

$$\theta_* = \frac{2\Omega}{k} \quad (k = 1, 2, 3, \ldots). \tag{1.32}$$

Equation (1.32) gives the relationship between the frequencies of the external force and the frequencies of the natural vibrations of the rod, in the vicinity of which the formation of unboundedly increasing oscillations is possible; namely, close to these relationships, the regions of the dynamic instability of a rod can be found.

We shall distinguish the first, second, third, etc., regions of dynamic instability according to the number k contained in Eq. (1.32). The region of instability situation near $\theta_* = 2\Omega$ is, as will be shown later, the most dangerous and has therefore the greatest practical importance. We will call this region the principal region of dynamic instability.

The origin of the resonance at $\theta = 2\Omega$ is easily seen from the following argument. Imagine that the rod (Fig. 2) oscillates in the transverse direction with the natural frequency Ω. During this oscillation, the longitudinal displacement of the moving end also will be a periodic function of time, having however the frequency 2Ω. Indeed for every period of transverse oscillation, two periods of oscillations of the moving support occur. To sustain the resonant oscillations so that the external force applied at the moving end has a frequency 2Ω, it is necessary that $\theta = 2\Omega$.

Before going on to further calculations, note the pecularity of this unique parametric resonance. If ordinary resonance of forced oscillations occurs during coincidence of the natural and exciting frequencies, then parametric resonance occurs during the coincidence of the exciting frequency with a doubled frequency of the natural oscillations. Another essential difference of parametric resonance lies in the possibility of exciting vibrations with frequencies smaller than the frequency of the principal resonance. Finally, qualitatively new in parametric resonance, is the existence of continuous regions of excitation (regions of dynamic instability), which we will now go on to calculate.

2. Since we are considering infinite determinants, the calculations can be expediently performed by systematically investigating the first, second, third and higher orders of the determinant. Hence the difference between two successive approximations serves as a practical estimation of the accuracy of computations.

For numerical calculations it is possible to represent the infinite determinants of the type in Eq. (1.28) in the form of chain fractions. We will show this on an example for the determinant

$$\begin{vmatrix} a_1 & 1 & 0 & 0 & \dots \\ 1 & a_2 & 1 & 0 & \dots \\ 0 & 1 & a_3 & 1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$

(any of our determinants can be reduced to such a form).

We will systematically expand the first, second and higher orders of the determinant. The equation of the first approximation evidently will be $a_1 = 0$. In the second approximation, we obtain

$$a_1 - \frac{1}{a_2} = 0.$$

The equation of the third approximation is

$$\begin{vmatrix} a_1 & 1 & 0 \\ 1 & a_2 & 1 \\ 0 & 1 & a_3 \end{vmatrix} = 0$$

which can be reduced to the form

$$a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3}} = 0 \; ;$$

and the general equation is represented by the form

$$a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{a_4 - \dots}}} = 0, \qquad (1.33)$$

Consider the determinant in Eq. (1.28), in which

$$a_1 = -\frac{1}{\mu}\left(1 \pm \mu - \frac{\theta^2}{4\Omega^2}\right),$$
$$a_k = -\frac{1}{\mu}\left[1 - \frac{(2k-1)^2\theta^2}{4\Omega^2}\right] \qquad (k \geqslant 2),$$

and Eq. (1.33) assumes the form

$$1 \pm \mu - \frac{\theta^2}{4\Omega^2} - \cfrac{\mu^2}{1 - \cfrac{9\theta^2}{4\Omega^2} - \cfrac{\mu^2}{1 - \cfrac{25\theta^2}{4\Omega^2} - \dots}} = 0,$$

or

$$\frac{\theta^2}{4\Omega^2} = 1 \pm \mu - \cfrac{\mu^2}{1 - \cfrac{9\theta^2}{4\Omega^2} - \cfrac{\mu^2}{1 - \cfrac{25\theta^2}{4\Omega^2} - \dots}}.$$

The formula obtained is especially convenient for the application of the method of successive approximations. Substituting an approximate value for the critical frequency on the right-hand side, we will obtain a more exact value each time.[6]

3. The advantage of the method described above is that it permits one to calculate the boundaries of the regions of instability with as high an accuracy as desired. At this point, we will not do any numerical calculations but will try to develop somewhat different formulas for the boundaries of the regions of instability.

We will examine Eq. (1.28) to determine the boundaries of the principal region of instability. Retaining the upper diagonal element, i.e., "determinant of first order," and equating it to zero

$$1 \pm \mu - \frac{\theta^2}{4\Omega^2} = 0,$$

we obtain the approximate formula for the boundaries of the principal region

$$\theta_* = 2\Omega\sqrt{1 \pm \mu}. \tag{1.34}$$

As is well-known, N. M. Beliaev derived the equation

$$\theta_* = 2\omega\sqrt{1 - \frac{P_0}{P_*}}\left[1 \pm \frac{P_t}{4(P_* - P_0)}\right],$$

which in our notations takes on the form

$$\theta_* = 2\Omega\left(1 \pm \frac{\mu}{2}\right).$$

[6] In this manner N. M. Beliaev, (Ref. 4) calculated the boundaries of the principal region of instability.

This formula was obtained by means of interpolation according to the results of separate numerical calculations of the Hill determinant, and can be considered sufficiently accurate. It is not difficult to see that up to the value $\mu = 0.5$, both equations give practically identical results.[7]

To increase the accuracy of Eq. (1.34), we will consider the second approximation

$$\begin{vmatrix} 1 \pm \mu - \frac{\theta^2}{4\Omega^2} & -\mu \\ -\mu & 1 - \frac{9\theta^2}{4\Omega^2} \end{vmatrix} = 0. \tag{1.35}$$

Substituting the approximate value of the critical frequency of Eq. (1.34) into the lower diagonal element of the determinant in Eq. (1.35), which affects the final results only slightly, and solving the equation with respect to θ, we obtain

$$\theta_* = 2\Omega \sqrt{1 \pm \mu + \frac{\mu^2}{8 \pm 9\mu}},$$

where the last term under the radical takes into account correction for the second approximation. This correction increases as μ increases, but even at $\mu = 0.3$ it does not exceed one percent. Thus, the accuracy of the very simple Eq. (1.34) is shown to be sufficient for practical purposes.

The result obtained is best understood if one remembers that "the determinant of first order" in Eq. (1.28) corresponds to taking into account the effect of the first terms of Eq. (1.27), i.e.,

$$f(t) = a \sin\frac{\theta t}{2} + b \cos\frac{\theta t}{2}.$$

The first approximation gives good results, signifying that the periodic solutions on the boundaries of the principal regions of instability are close to harmonic vibrations. We will return to this deduction later on.

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[7] This value as follows from Eq. (1.9) corresponds to the case which is hardly reached in practical problems: $P_0 + P_t = P_*$

Let us dwell upon one interpretation of Eq. (1.34). Rewriting this equation in the form

$$\theta_* = 2\omega\sqrt{1 - \frac{P_0}{P_*} \pm \frac{P_t}{2P_*}}$$

we can compare it with Eq. (1.8), which determines the natural frequency of a rod loaded by a constant axial force

$$\Omega = \omega\sqrt{1 - \frac{P_0}{P_*}}.$$

Comparing these equations, we can arrive at the conclusion that the frequencies corresponding to the boundary of the principal region of dynamic instability for the first approximation can be determined as the doubled frequencies of the natural oscillations of the rod loaded with the constant longitudinal forces $P_o + 1/2\,P_t$ and $P_o - 1/2P_t$, respectively.

To determine the boundaries of the second region of instability, it is necessary to consider Eqs. (1.29) and (1.30). By restricting ourselves to determinants of second order

$$\begin{vmatrix} 1 - \frac{\theta^2}{\Omega^2} & -\mu \\ -\mu & 1 - \frac{4\theta^2}{\Omega^2} \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & -\mu \\ -2\mu & 1 - \frac{\theta^2}{\Omega^2} \end{vmatrix} = 0,$$

we obtain the following approximate formulas for the critical frequencies

$$\left.\begin{aligned} \theta_* &= \Omega\sqrt{1 + \tfrac{1}{3}\mu^2}, \\ \theta_* &= \Omega\sqrt{1 - 2\mu^2}. \end{aligned}\right\} \qquad (1.36)$$

These equations can be made more accurate if one considers determinants of higher order.[8]

To calculate the third region of instability, one must refer to Eq. (1.28). Thus, proceeding from the determinant of second order, Eq. (1.35), one obtains

$$\theta_{*} = \frac{2}{3}\Omega\sqrt{1 - \frac{9\mu^2}{8 \pm 9\mu}}. \tag{1.37}$$

Comparing Eqs. (1.34), (1.36), and (1.37), we see that the width of the regions of dynamic instability rapidly decreases as the number of the region increases

$$\frac{\Delta\theta}{\Omega} \sim \mu,\ \mu^2,\ \mu^3,\ \ldots \tag{1.38}$$

The principal region of instability has the greatest width.

The distribution of the first three regions of instability on the plane $(\mu, \theta/2\Omega)$ is shown in Fig. 5 (the regions of instability are crosshatched). In contrast to Fig. 3, the values $\lambda^{-1/2}$ are plotted here on the vertical axis and the vaules $h^2/2$ on the horizontal axis. In addition to this, Fig. 5 considers only that part of the plane of the changed parameters which is of practical interest. This part of the plane is surrounded by a frame in Fig. 3.

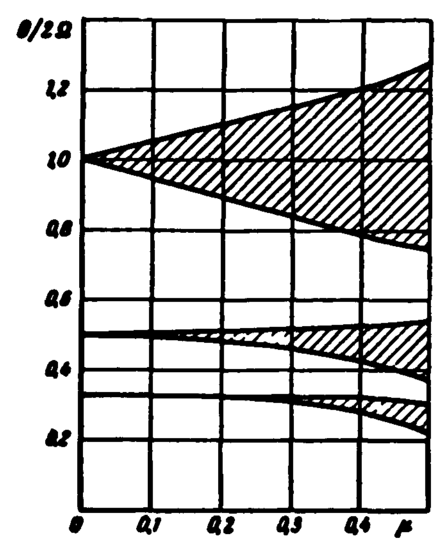

Figure 5

[8] In the literature (see for example Chelomei, Ref. 17), one can find the assertion that the second, and generally the even regions of instability degenerate into curves. This assertion is incorrect; the source of the error is in the poor choice of the zero approximation in "the method of the small parameter."

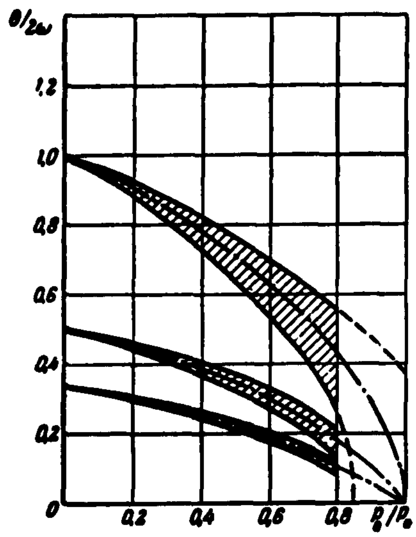

Figure 6

The regions of instability on the plane $(P_o/P_*, \theta/2\Omega)$ appear in Fig. 6. The ratio P_t/P_o is constant and is equal to 0.3. As $P_t \to 0$, the regions of instability degenerate into the backbone curves given by

$$\frac{\theta}{2\omega} = \frac{1}{k}\sqrt{1 - \frac{P_0}{P_*}} \quad (k = 1, 2, 3, \ldots).$$

The boundaries of the regions cannot be obtained from Eqs. (1.34), (1.36), and the other equations when P_o/P_* is large. Here one must use the Mathieu diagram (Fig. 3) or tables of the eigenvalues of the Mathieu equation.[9]

4. We shall briefly dwell upon the application of the method stated above to the more general case of the Hill equation. Let the longitudinal force change according to the law

$$P(t) = P_0 + \sum_{k=1}^{\infty} P_{tk} \cos k\theta t.$$

The corresponding equation will be

$$f'' + \Omega^2 \left(1 - \sum_{k=1}^{\infty} 2\mu_k \cos k\theta t\right) f = 0, \tag{1.39}$$

where

$$\mu_k = \frac{P_{tk}}{2(P_* - P_0)}.$$

[9]See Ref. 73. The construction of the regions of instability for the case of large coefficients of excitation was done by Lubkin and Stoker (Ref. 11).

Again we seek the periodic solution in series form

$$f(t)=\sum_{k=1,3,5}^{\infty}\left(a_k \sin\frac{k\theta t}{2}+b_k\cos\frac{k\theta t}{2}\right),$$

substituting this equation into Eq. (1.39), we obtain, for the critical frequencies:

$$\begin{vmatrix} 1\pm\mu_1-\frac{\theta^2}{4\Omega^2} & -(\mu_1\pm\mu_2) & -(\mu_2\pm\mu_3)\dots \\ -(\mu_1\pm\mu_2) & 1\pm\mu_3-\frac{9\theta^2}{4\Omega^2} & -(\mu_1\pm\mu_4)\dots \\ -(\mu_2\pm\mu_3) & -(\mu_1\pm\mu_4) & 1\pm\mu_5-\frac{25\theta^2}{4\Omega^2}\dots \\ \dots & \dots & \dots \end{vmatrix}=0. \tag{1.40}$$

The rest of the equations have an analogous form. Retaining in them the diagonal elements, i.e., neglecting in the final computations the influence of the harmonics, we obtain

$$\theta_*\approx\frac{2\Omega}{k}\sqrt{1\pm\mu_k}\qquad (k=1,\ 2,\ 3,\dots). \tag{1.41}$$

Comparing this equation with Eq. (1.34), which corresponds to the case of a harmonically changing longitudinal force, we see that in the first approximation, each region of instability depends only on the corresponding harmonic in the expansion of the longitudinal force. This case was observed already by N. M. Krylov and N. N. Bogoliubov (Ref. 5). The influence of the ith harmonic on the width of the kth region of instability comprises the magnitude of the order $(\mu_k \pm \mu_i)^2$, as is seen from the equation of the critical frequencies.

We will apply the results obtained to the case when the longitudinal force changes according to a piecewise-constant law (•3). Having dis-

placed the initial instant of time by T/4, we present the expression for the longitudinal force in the form of the trigometric series

$$P(t) = P_0 + \frac{4P_t}{\pi} \sum_{k=1,3}^{\infty} \frac{1}{k} \cos k\theta t,$$

which converges uniformly everywhere with the exception of the points at which the magnitude of the longitudinal force changes. We obtain

$$\mu_k = \frac{4}{\pi k} \mu,$$

where

$$\mu = \frac{P_t}{2(P_* - P_0)}.$$

For this case Eq. (1.41) gives

$$\theta_* = \frac{2\Omega}{k} \sqrt{1 \pm \frac{4}{\pi k} \mu} \qquad (k = 1, 3, 5, \ldots). \tag{1.42}$$

Comparing Eq. (1.34) with the equation obtained for k = 1, we arrive at the conclusion that, in the case of the rectangular variation of longitudinal force, the principal region of instability appears to be approximately $4/\pi$ times wider than that described by the Mathieu equation.

The harmonic longitudinal force is sometimes replaced by a force changing according to the piecewise-constant law, its amplitude being determined from some other prior consideration. This replacement can be justified when discussing the principal region of instability. Qualitatively incorrect results are obtained, however, if the secondary regions are converted to a piecewise-constant law of changing longitudinal force. Thus, the third, fifth and, generally the odd regions of instability for the case of a

harmonic longitudinal force have a width of the order $\Delta\theta/\Omega \sim \mu^k$, but in the case of piecewise-constant law the width is of the order $\Delta\theta/\Omega \sim \mu/k$.

•6. SOME EXPERIMENTAL RESULTS

The experimental verification of the theoretical results presented above can easily be carried out in the laboratory (Ref. 56). One of the possible designs of the experimental setup is shown in Fig. 7.

The test fixture is assembled on the base of a vertical impact pile-driver. The specimen (No. 3 on diagram) made of flat-bar steel is placed between the guides of the pile-driver (No. 4), while the stationary support of the specimen is fixed on the lower plate and the moving support slides in the guide. This installation provides free vertical translation of the moving end of the rod; free rotation of the support cross sections is provided by means of ball bearings. All this allows the conditions of the test to approach the theoretical conditions.

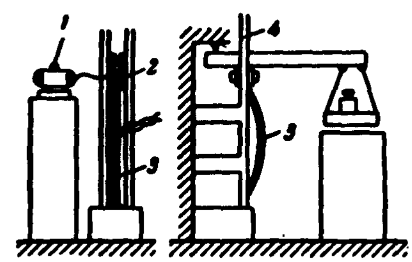

Figure 7

In the author's experiments, strain-gages in conjunction with a loop oscillograph registered and recorded the oscillations. To eliminate the deformations due to axial compression, two gages were used, one of which was posted on the tension side and another on the compression side. The gages were connected in parallel to the circuit of a measuring bridge, i.e., in the two adjacent arms so that the influence of the deformation of the opposite sign is doubled and eliminates the compression deformation.

The gages (constant) had the sensitivity $s = 2.1$ and a resistance of 200 Ω. An ac amplifier was used, with a carrier frequency of 8000 cps. A type B-4 vibrator was used. The characteristics of the vibration were: sensitivity was 1 mm/ma, natural frequency of the system in air 3500 cps, resistance 1 Ω, and maximum current 100 ma.

The eccentric vibrator (No. 2) produced the periodic component of the longitudinal force. The frequency of the load was determined by a sliding contact on the shaft of the vibrator, periodically closing a circuit connected in the system. This simple device made it possible to determine not only the phase time, but also the phase angle between the external force and the excited vibrations.

The experiments confirmed the theoretical conclusion concerning the existence of a continuous region of dynamic instability. Generally, the periodic longitudinal force induces transverse oscillations at any frequency. The amplitude of these oscillations is negligible, and the oscillations take place with the frequency of the external force. These oscillations are obviously dependent on the initial eccentricity of the longitudinal force. However, in a certain range of frequencies lying in the vicinity of $\theta = 2\Omega$, strong transverse oscillations develop with amplitudes increasing to high values.

Characteristically, this growth, at least initially, follows the exponential law (Fig. 8). This is in complete agreement with theoretical results, according to which the solution of the Mathieu equations on the boundaries of the regions of instability have the form

$$f(t) = \chi(t)\, e^{\frac{t}{T}\ln\rho},$$

where $\ln\rho$ is a real quantity.

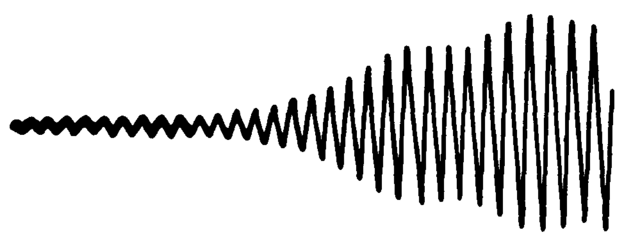

Figure 8

At this point, it is worth mentioning the experiments concerning the parametric excitation of electrical oscillations conducted under the guidance of L. I. Mandel'shtam and N. D. Papalski (Ref. 78). Figure 9 shows an oscillogram of the growth of the current in an oscillatory contour, the inductance of which periodically changes with time by means of the aid of an external mechanical force (Ref. 79). The character of this oscillogram if a perfect analogue to our oscillograms. In particular, in both cases the amplitude growth follows an exponential law. Further growth of amplitude is slowed down and finally stops, which in both cases is caused by the effect of non-linear factors. This question will be examined in detail somewhat later.

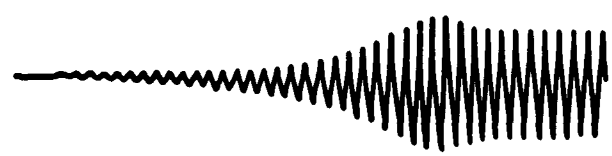

Figure 9

The experiment confirms not only the qualitative but the quantitative results, and in particular, Eq. (1.34). In using this formula, however, one must take into account that the force developed in the vibrator increases in proportion to the square of the frequency. It is possible to write down that

$$P_t = \frac{\theta^2}{4\Omega^2}\bar{P}_t,$$

where $\bar{P}_t$ is the amplitude of the longitudinal force which the vibrator develops at the frequency 2Ω. By considering this relation, and from Eq. (1.34), we obtain

$$\theta_* = \frac{2\Omega}{\sqrt{1 \pm \frac{\bar{P}_t}{2(P_* - P_0)}}},$$

or the shorter form

$$\theta_{*} = \frac{2\Omega}{\sqrt{1 \pm \mu}}. \tag{1.43}$$

The principal region of instability, the boundaries of which are determined from Eq. (1.43) is shown in Fig. 10. On the same curve, the experimental results are plotted, which obviously agree well with theory. The boundaries of the principal region of instability emerge very precisely at the appearance of oscillations occurring with a frequency less than twice that of the external forces, can be established with high accuracy. Outside the region of instability, as already mentioned, the steady-state oscillations occur at the frequency of the external load (Fig. 11).

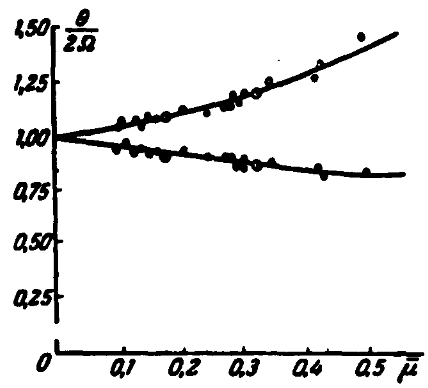

Figure 10

The above results are related to the principal region of instability. No correlation could be found experimentally regarding the second, third, and higher number regions—at least for a small amplitude of longitudinal force.[10] It is true that in approaching the synchronism $\theta = \Omega$, the intensity of oscillations occurring with a given external force

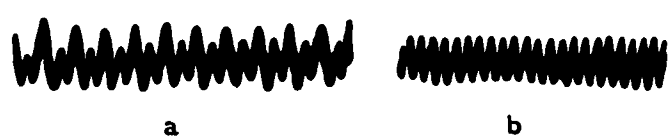

Figure 11

amplitude increases somewhat. However, the region of increase in vibration amplitude is not bounded by some narrow band of frequencies, but spreads

[10] Translator's note; These secondary regions were found experimentally by Utida and Sezawa (Ref. 10) and Weingarten (Ref. 80*)

far beyond the limits of the boundaries predicted from theory. This result gives a basis for supposing that similar oscillations are produced by the influence of additional factors such as eccentricity and initial curvature. As we will see in the following sections, this lack of agreement between theory and experiment for the secondary regions of instability can be removed by considering the problem with damping.

CHAPTER TWO

THE INFLUENCE OF DAMPING ON THE REGIONS OF DYNAMIC INSTABILITY

•7. INVESTIGATION OF THE DIFFERENTIAL EQUATIONS

1. For reasons which will be understood by the reader later on (• 11), we will restrict ourselves here to the effect of linear damping. More precisely, we will consider forces of resistance which introduce into the differential equation an additional term with a first derivative of the displacement with respect to time:

$$f'' + 2\epsilon f' + \Omega^2(1 - 2\mu\cos\theta t)f = 0. \tag{2.1}$$

The coefficient of damping ϵ will be determined experimentally for each case.

We write the solution of the differential equation, Eq. (2.1), in the form

$$f(t) = u(t)\cdot v(t),$$

where u(t) and v(t) are at present unknown functions of time. A substitution in Eq. (2.1) gives:

$$u''v + 2u'(v' + \epsilon v) + \Omega^2(1 - 2\mu\cos\theta t)uv + uv'' + 2\epsilon uv' = 0$$

We will require the coefficient of u' to vanish in the above expression. In this way we arrive at the following two differential equations

$$\begin{gathered} u''v + \Omega^2(1 - 2\mu\cos\theta t)uv + uv'' + 2\epsilon uv' = 0, \\ v' + \epsilon v = 0. \end{gathered}$$

The second equation gives $v = Ce^{-\epsilon t}$; after substituting back in the first equation and dividing by $Ce^{-\epsilon t}$, we obtain

$$u'' + \Omega^2\left(1 - \frac{\epsilon^2}{\Omega^2} - 2\mu\cos\theta t\right)u = 0. \tag{2.2}$$

The Mathieu-Hill equation thus obtained differs from the equation of the conservative problem, Eq. (1.10), by the presence of an additional damping term which represents a correction to the frequency

$$\Omega_\epsilon = \Omega\sqrt{1 - \frac{\epsilon^2}{\Omega^2}}. \tag{2.3}$$

As already shown in Eq. (1.21), the two linear independent solutions of the Mathieu-Hill equation have the form

$$u_1(t) = \chi_1(t)e^{\frac{t}{T}\ln\rho_1},$$
$$u_2(t) = \chi_2(t)e^{\frac{t}{T}\ln\rho_2},$$

where $\chi_{1,2}(t)$ are periodic functions of period T, and $\rho_{1,2}$ are roots of the characteristic equation. These roots are connected by the relationship in Eq. (1.20)

$$\rho_1 \cdot \rho_2 = 1.$$

Returning to Eq. (2.1), we can represent its solution in the form

$$f_1(t) = \chi_1(t)\exp\left(\frac{t}{T}\ln\rho_1 - \epsilon t\right),$$
$$f_2(t) = \chi_2(t)\exp\left(\frac{t}{T}\ln\rho_2 - \epsilon t\right),$$

or, isolating the real part of ln ρ, in the form

$$\left.\begin{aligned} f_1(t) &= \varphi_1(t)\exp\left(\frac{t}{T}\ln|\rho_1| - \epsilon t\right), \\ f_2(t) &= \varphi_2(t)\exp\left(\frac{t}{T}\ln|\rho_2| - \epsilon t\right). \end{aligned}\right\} \tag{2.4}$$

Here, as before, $\phi_{1,2}(t)$ are bounded (almost periodic) functions

$$\varphi_1(t) = \chi_1(t)e^{\frac{it}{T}\arg\rho_1},$$
$$\varphi_2(t) = \chi_2(t)e^{\frac{it}{T}\arg\rho_2}.$$

2. One easily sees that the behavior of the solutions of Eq. (2.1) depends on the relationship between the coefficient of damping ϵ and the real part of ln ρ. This means that the solutions will unboundedly increase when

$$\epsilon < \frac{\ln|\rho|}{T}$$

and will be damped when

$$\epsilon > \frac{\ln|\rho|}{T}.$$

In examining this question in more detail, let the characteristic numbers $\rho_{1,2}$ be complex conjugates; then,

$$\ln|\rho| = 0,$$

and both solutions lead to vibrations which are as quickly damped as the corresponding free vibrations:

$$f_1(t) = \varphi_1(t)e^{-\epsilon t},$$
$$f_2(t) = \varphi_2(t)e^{-\epsilon t}.$$

We will now investigate the cases of real roots; moreover, we shall assume

$$|\rho_1| < 1, \quad |\rho_2| > 1.$$

Then the first solution will be damped with time; for the second solution,

$$f_2(t) = \varphi_2(t) \exp\left(\frac{t}{T} \ln \rho_2 - \varepsilon t\right),$$

two cases must be investigated. If

$$\varepsilon > \frac{\ln|\rho_2|}{T},$$

then the second solution will be damped. If, however,

$$\varepsilon < \frac{\ln|\rho_2|}{T},$$

then the second solution, and consequently the general integral, will unboundedly increase with time.

We will investigate the boundary case

$$\varepsilon = \frac{\ln|\rho_2|}{T}.$$

It is of special importance to us that the second solution be periodic: namely, at $\rho > 0$ a period T will occur, and at $\rho < 0$, a period 2T.

Thus, the problem of finding the regions of instability for Eq. (2.1) is reduced to the determination of the conditions under which it has periodic solutions with periods T and 2T. Here, also, two solutions of an identical period bound the region of increasing solutions and two solutions of different periods bound the region of damped solutions.

Note that the regions of instability for Eq. (2.1) lie inside the regions of instability for Eq. (2.2). The latter describes, by the way, the vibrations of a conservative system with a frequency calculated with corrections for damping.

•8. DERIVATION OF THE EQUATION OF THE CRITICAL FREQUENCIES INCLUDING THE CONSIDERATION OF DAMPING

Further calculations are not difficult. For determining the conditions under which Eq. (2.1) has periodic solutions with period 2T, we substitute into this equation the series[1]

$$f(t)=\sum_{k=1,3,5}^{\infty}\left(a_k\sin\frac{k\theta t}{2}+b_k\cos\frac{k\theta t}{2}\right)$$

and carrying out the trigometric transformations, we then equate the coefficients of the same $\sin (k\theta t)/2$ and $\cos (k\theta t)/2$. As a result, we obtain the system of linear algebraic equations:

$$\left.\begin{aligned}
\left(1+\mu-\frac{\theta^2}{4\Omega^2}\right)a_1-\mu a_3-\frac{\Delta}{\pi}\frac{\theta}{2\Omega}b_1=0,\\
\left(1-\mu-\frac{\theta^2}{4\Omega^2}\right)b_1-\mu b_3+\frac{\Delta}{\pi}\frac{\theta}{2\Omega}a_1=0,\\
\left(1-\frac{k^2\theta^2}{4\Omega^2}\right)a_k-\mu(a_{k-2}+a_{k+2})-\frac{\Delta}{\pi}\frac{k\theta}{2\Omega}b_k=0,\\
\left(1-\frac{k^2\theta^2}{4\Omega^2}\right)b_k-\mu(b_{k-2}+b_{k+2})+\frac{\Delta}{\pi}\frac{k\theta}{2\Omega}a_k=0\\
(k=3,\ 5,\ \ldots),
\end{aligned}\right\} \tag{2.5}$$

where Δ denotes the decrement of damping of the natural oscillations of a rod loaded by a constant component of longitudinal force

$$\Delta=\frac{2\pi\varepsilon}{\omega\sqrt{1-\frac{P_0}{P_*}}}. \tag{2.6}$$

[1] This method was applied by Rayleigh (Ref. 1), for investigating the conditions necessary for "sustaining motion." See also Ref. 29.

Equating the determinant of the homogeneous system Eq. (2.5) to zero, we obtain an equation for the critical frequencies

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ 1-\frac{9\theta^2}{4\Omega^2} & -\mu & 0 & -\frac{\Delta}{\pi}\frac{3\theta}{2\Omega} \\ -\mu & 1+\mu-\frac{\theta^2}{4\Omega^2} & -\frac{\Delta}{\pi}\frac{\theta}{2\Omega} & 0 \\ 0 & \frac{\Delta}{\pi}\frac{\theta}{2\Omega} & 1-\mu-\frac{\theta^2}{4\Omega^2} & -\mu \\ \frac{\Delta}{\pi}\frac{3\theta}{2\Omega} & 0 & -\mu & 1-\frac{9\theta^2}{4\Omega^2} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \tag{2.7}$$

This equation makes it possible to calculate the boundaries of the region of instability which lie near the frequency

$$\theta_* = \frac{2\Omega}{k} \qquad (k = 1,\ 3,\ 5,\ \ldots).$$

The second equation is obtained by taking a solution in the form of the series

$$f(t) = b_0 + \sum_{k=2,4,6}^{\infty}\left(a_k \sin\frac{k\theta t}{2} + b_k \cos\frac{k\theta t}{2}\right).$$

A substitution of this series leads to a system of equations

$$\left.\begin{aligned} b_0 - \mu b_2 = 0, \\ \left(1-\frac{\theta^2}{\Omega^2}\right)a_2 - \mu a_4 - \frac{\Delta}{\pi}\frac{\theta}{\Omega}b_2 = 0, \\ \left(1-\frac{\theta^2}{\Omega^2}\right)b_2 - \mu(2b_0 + b_4) + \frac{\Delta}{\pi}\frac{\theta}{\Omega}a_2 = 0, \\ \left(1-\frac{k^2\theta^2}{4\Omega^2}\right)a_k - \mu(a_{k-2} + a_{k+2}) - \frac{\Delta}{\pi}\frac{k\theta}{2\Omega}b_k = 0, \\ \left(1-\frac{k^2\theta^2}{4\Omega^2}\right)b_k - \mu(b_{k-2} + b_{k+2}) + \frac{\Delta}{\pi}\frac{k\theta}{2\Omega}a_k = 0 \\ (k = 4, 6, \ldots). \end{aligned}\right\} \tag{2.8}$$

The equation of the critical frequencies

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ 1-\frac{4\theta^2}{\Omega^2} & -\mu & 0 & 0 & -\frac{\Delta}{\pi}\frac{2\theta}{\Omega} \\ -\mu & 1-\frac{\theta^2}{\Omega^2} & 0 & -\frac{\Delta}{\pi}\frac{\theta}{\Omega} & 0 \\ 0 & 0 & 1 & -\mu & 0 \\ 0 & \frac{\Delta}{\pi}\frac{\theta}{\Omega} & -2\mu & 1-\frac{\theta^2}{\Omega^2} & -\mu \\ \frac{\Delta}{\pi}\frac{2\theta}{\Omega} & 0 & 0 & -\mu & 1-\frac{4\theta^2}{\Omega^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \tag{2.9}$$

makes it possible to find the regions of instability which lie near

$$\theta_* = \frac{2\Omega}{k} \qquad (k = 2, 4, 6, \ldots).$$

It is easy to see that at $\Delta = 0$ the equations obtained coincide with the equations of the conservative problem, Eqs. (1.28), (1.29) and (1.30).

Furthermore, let us consider the case when $\mu \to 0$ which corresponds to an infinitely small amplitude of the longitudinal force. The determinant, Eq. (2.7), takes the form

$$\Delta(\theta) = \Delta_1(\theta)\cdot\Delta_3(\theta)\ldots\Delta_k(\theta)\ldots, \tag{2.10}$$

for this case, where

$$\Delta_k(\theta) = \begin{vmatrix} 1-\frac{k^2\theta^2}{4\Omega^2} & -\frac{\Delta}{\pi}\frac{k\theta}{2\Omega} \\ \frac{\Delta}{\pi}\frac{k\theta}{2\Omega} & 1-\frac{k^2\theta^2}{4\Omega^2} \end{vmatrix}.$$

One can write the determinant, Eq. (2.9), in an analogous form

$$\Delta(\theta) = \Delta_2(\theta)\cdot\Delta_4(\theta)\ldots\Delta_k(\theta)\ldots, \tag{2.11}$$

where $\Delta_k(\theta)$ is defined as before.

Since all $\Delta_k(\theta) > 0$, the determinants in Eqs. (2.10) and (2.11) cannot take zero values. Owing to the uniformity of the determinant, the determinants on the left-hand side of Eq. (27) and, respectively, Eq. (29) are not equal to zero for a sufficiently small value of the exciting parameter. In other words, in the presence of damping, the loss of dynamic stability of the straight form of the rod can occur only at values of the amplitude of the longitudinal force greater than a certain minimum value.

The determination of these values (which we will call critical later on) represents general practical interest.

•9. DETERMINATION OF THE CRITICAL VALUES OF THE COEFFICIENT OF EXCITATION

1. We begin with the principal boundary of the region of instability, for which purpose we retain in the determinant, Eq. (2.7), the central elements.

$$\begin{vmatrix} 1+\mu-\frac{\theta^2}{4\Omega^2} & -\frac{\Delta}{\pi}\frac{\theta}{2\Omega} \\ \frac{\Delta}{\pi}\frac{\theta}{2\Omega} & 1-\mu-\frac{\theta^2}{4\Omega^2} \end{vmatrix} = 0. \qquad (2.12)$$

Solving Eq. (2.12) with respect to the excitation frequency, we obtain

$$\theta_* = 2\Omega\sqrt{1-\frac{1}{2}\left(\frac{\Delta}{\pi}\right)^2 \pm \sqrt{\mu^2-\left(\frac{\Delta}{\pi}\right)^2+\frac{1}{4}\left(\frac{\Delta}{\pi}\right)^4}}.$$

Since the decrement of damping Δ is usually very small compared to unity (Δ = 0.01 - 0.05), we can simplify the formula obtained by neglecting the terms containing higher powers of Δ/π:

$$\theta_* = 2\Omega\sqrt{1 \pm \sqrt{\mu^2-\left(\frac{\Delta}{\pi}\right)^2}}. \qquad (2.13)$$

We will investigate Eq. (2.13). As long as the expression under the inner radical is positive, for the critical frequency this formula gives two real values which correspond to two boundaries of the principal region of

instability. The limiting case (Figure 12)

$$\mu^2 - \left(\frac{\Delta}{\pi}\right)^2 = 0$$

determines the minimum value of the coefficient of excitation for which the occurrence of undamped oscillations is still possible. Thus, the critical value of the coefficient of excitation is

$$\mu_{*1} = \frac{\Delta}{\pi}. \qquad (2.14)$$

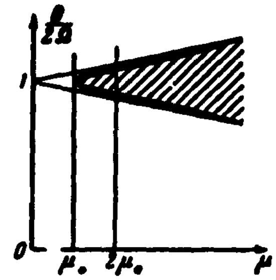

Figure 12

Equation (2.14) shows that the greater the damping, the greater the longitudinal force required to cause dynamic instability of the rod. Note that the influence of damping manifests itself to a noticeable degree only for small coefficients of excitation. Therefore, the boundaries of the region of instability determined according to Eqs. (1.34) and (2.13) for $\mu > 2\mu_*$ practically coincide.

2. We shall proceed now to the determination of the boundaries of the secondary regions of instability. We equate the determinant composed of the central elements of the determinant in Eq. (2.9) to zero:

$$\begin{vmatrix} 1 - \frac{\theta^2}{\Omega^2} & 0 & -\frac{\Delta}{\pi}\frac{\theta}{\Omega} \\ 0 & 1 & -\mu \\ \frac{\Delta}{\pi}\frac{\theta}{\Omega} & -2\mu & 1 - \frac{\theta^2}{\Omega^2} \end{vmatrix} = 0.$$

The solution of this equation gives

$$\theta_* = \Omega\sqrt{1 - \mu^2 \pm \sqrt{\mu^4 - \left(\frac{\Delta}{\pi}\right)^2 (1 - \mu^2)}}. \qquad (2.15)$$

The minimum value μ for which Eq. (2.15) gives two real values for the frequency is found from the condition

$$\mu^4 - \left(\frac{\Delta}{\pi}\right)^2 (1 - \mu^2) = 0.$$

Solving this equation, we find approximately

$$\mu_{*3} = \sqrt{\frac{\Delta}{\pi}}. \tag{2.16}$$

For determining the boundaries of the third region of instability we return to Eq. (2.7), retaining it in all the written elements. The exact solution of such an equation is difficult, therefore, we will substitute the approximate value of the critical frequency $\theta_* = (2\Omega)/3$ in all the elements, which only slightly influences the final result, i.e., in all except the upper and lower diagonal elements. Equation (2.7) can be rewritten then in the form

$$\begin{vmatrix} \xi & -\mu & 0 & -\frac{\Delta}{\pi} \\ 0 & \frac{8}{9}+\mu & -\frac{\Delta}{3\pi} & 0 \\ 0 & \frac{\Delta}{3\pi} & \frac{8}{9}-\mu & -\mu \\ \frac{\Delta}{\pi} & 0 & -\mu & \xi \end{vmatrix} = 0,$$

where, for simplification, we let

$$\xi = 1 - \frac{9\theta^2}{4\Omega^2}.$$

Resolving the determinant and neglecting magnitudes of order $(\Delta/\pi)^4$, $(\Delta/\pi)^6$, etc., we obtain

$$\xi = \frac{\frac{8}{9}\mu^3 \pm \sqrt{\mu^6 - \left(\frac{\Delta}{\pi}\right)^2\left(\frac{64}{81} - \frac{2}{3}\mu^2\right)}}{\frac{64}{81} - \mu^2}. \tag{2.17}$$

The critical frequency is calculated according to the equation

$$\theta_* = \frac{2\Omega}{3}\sqrt{1-\xi}. \tag{2.18}$$

where ξ is determined from Eq. (2.17). One can see from Eqs. (2.17) and (2.18) that the third resonance occurs only when

$$\frac{\Delta}{\pi} < \frac{\mu^3}{\frac{64}{81} - \frac{2}{3}\mu^2},$$

which gives approximately

$$\mu_{*3} = \sqrt[3]{\frac{\Delta}{\pi}}. \tag{2.19}$$

Combining Eqs. (2.14), (2.16) and (2.19), we arrive at the conclusion that for the excitation of oscillations at the critical frequency of the kth order,

$$\theta_* = \frac{2\Omega}{k} \qquad (k = 1, 2, 3, \ldots),$$

it is necessary that the coefficient of excitation exceed the critical value

$$\mu_{*k} = \sqrt[k]{\frac{\Delta}{\pi}} \qquad (k = 1, 2, 3, \ldots). \tag{2.20}$$

3. Now one can finally answer the question as to why the principal region of instability is the most critical.

A graph of the distribution of the regions of instability including damping is presented in Fig. 13. The graph differs from the corresponding graph for the conservation problem (Figure 5). The presence of damping cuts off that part of the regions of instability which border on the axis of the ordinate, and makes impossible the onset of resonance for sufficiently small coefficients of excitation. It is interesting to note that the effect of damping, which is not essential for the principal region of instability, becomes particularly noticeable with respect to the secondary regions. This is seen not only from Figure 13 but from Figure 14 as well, where the dependence of the critical coefficient of excitation of the damped rod is shown.

For example, for a decrement of damping $\Delta = 0.01$, the lowest value of the coefficient of excitation at which principal resonance can still occur is $\mu_{*1} = 0.0032$. In other words, principal resonance can be realized with an amplitude P_t of the periodic force which is less than one percent of the Euler value. For the second resonance we obtain $\mu_{*2} = 0.057$, i.e., a value seventeen times greater, which corresponds to a periodic force amplitude P_t, approximately 12 percent of the Euler value. Still larger longitudinal forces are required in order to excite the third, fourth, and higher orders of resonances. Such values of the coefficient of excitation are rarely found in engineering practice.

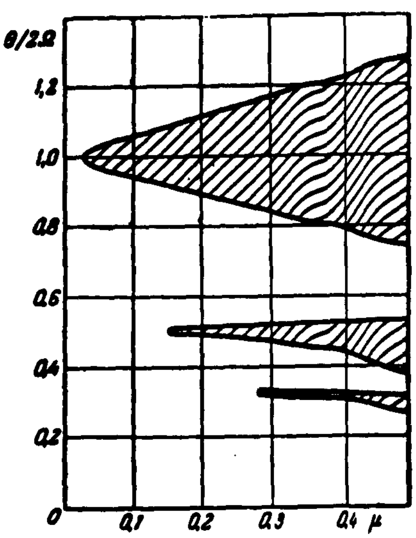

Figure 13

The considerations mentioned above show what an important role damping plays in problems of dynamic stability of elastic systems. Unfortunately a systematic study of the damping of engineering structures is not available. More studies have been made on that part of damping which is related to the dissipation of energy in the material of vibrating structures. But, even in machine elements where the character of the work tends to reduce the external loss of energy to a minimum, the internal dissipation of energy comprises only a small part of the general losses. In structures, the role of the external loss (the loss in the supports and couplings, and the loss in the environment) is undoubtedly much greater. Therefore, it is only possible to approximately indicate the limits of the variation of the decrements of damping (for steel constructions $\Delta = 0.005 - 0.05$).

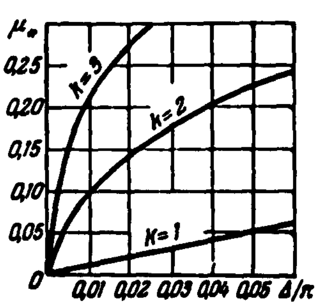

Figure 14

In any case, the above analysis shows that the principal region of instability is the most critical; the second and, even more so, the third regions of instability can be realized only with sufficiently large amplitudes of the longitudinal force.

•10. THE GENERAL CASE OF ANY PERIODIC LOAD

Above, we considered in detail the case of a harmonically varying longitudinal force. For the more general case of periodic loading, we shall confine ourselves to brief remarks.

It is evident that all the arguments mentioned in •7 are also valid for the case when the loading is given in the form of the series

$$P(t)=P_0+\sum_{k=1}^{\infty} P_{tk}\cos k\theta t.$$

The differential equation of the problem will be

$$f''+2\varepsilon f'+\Omega^2\left(1-\sum_{k=1}^{\infty} 2\mu_k \cos k\theta t\right)f=0, \qquad (2.21)$$

where

$$\mu_k=\frac{P_{tk}}{2(P_*-P_0)}.$$

The periodic solutions of Eq. (2.21) correspond as before to the boundaries of the regions of dynamic instability. Letting

$$f(t)=\sum_{k=1,3,5}^{\infty}\left(a_k\sin\frac{k\theta t}{2}+b_k\cos\frac{k\theta t}{2}\right),$$

we arrive at the equation of the critical frequencies

$$\left|\begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ 1+\mu_3-\frac{9\theta^2}{4\Omega^2} & -(\mu_1+\mu_2) & 0 & -\frac{\Delta}{\pi}\frac{3\theta}{2\Omega} \\ -(\mu_1+\mu_2) & 1+\mu_1-\frac{\theta^2}{4\Omega^2} & -\frac{\Delta}{\pi}\frac{\theta}{2\Omega} & 0 \\ 0 & \frac{\Delta}{\pi}\frac{\theta}{2\Omega} & 1-\mu_1-\frac{\theta^2}{4\Omega^2} & -(\mu_1-\mu_2) \\ \frac{\Delta}{\pi}\frac{3\theta}{2\Omega} & 0 & -(\mu_1-\mu_2) & 1-\mu_3-\frac{9\theta^2}{4\Omega^2} \\ \cdots & \cdots & \cdots & \cdots \end{array}\right| = 0. \tag{2.22}$$

This equation makes it possible to calculate the boundaries of all the odd regions of instability. For the even regions of instability we obtain

$$\left|\begin{array}{ccccc} \cdots & \cdots & \cdots & \cdots & \cdots \\ 1+\mu_4-\frac{4\theta^2}{\Omega^2} & -(\mu_1+\mu_3) & 0 & 0 & -\frac{\Delta}{\pi}\frac{2\theta}{\Omega} \\ -(\mu_1+\mu_3) & 1+\mu_2-\frac{\theta^2}{\Omega^2} & 0 & -\frac{\Delta}{\pi}\frac{\theta}{\Omega} & 0 \\ 0 & 0 & 1 & -\mu_1 & 0 \\ 0 & \frac{\Delta}{\pi}\frac{\theta}{\Omega} & -2\mu_1 & 1-\mu_2-\frac{\theta^2}{\Omega^2} & -(\mu_1-\mu_3) \\ \frac{\Delta}{\pi}\frac{2\theta}{\Omega} & 0 & 0 & -(\mu_1-\mu_3) & 1-\mu_4-\frac{4\theta^2}{\Omega^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right| = 0. \tag{2.23}$$

In the first approximation we try to neglect the mutual effect of separate harmonics in the expansion of the longitudinal force. The determinants of Eq. (2.22) and 2.23) break down into separate equations

$$\left|\begin{array}{cc} 1+\mu_k-\frac{k^2\theta^2}{4\Omega^2} & -\frac{\Delta}{\pi}\frac{k\theta}{\Omega} \\ \frac{\Delta}{\pi}\frac{k\theta}{\Omega} & 1-\mu_k-\frac{k^2\theta^2}{4\Omega^2} \end{array}\right| = 0 \quad (k=1, 2, 3, \ldots). \tag{2.24}$$

From Eq. (2. 24) one can find a relationship between the parameters which is necessary to excite the first, second, and higher resonances:

$$\mu_k > \frac{\Delta}{\pi} \qquad (k = 1,\ 2,\ 3,\ \ldots). \tag{2.25}$$

According to this formula, the formation of the kth resonance depends solely on the kth harmonic of the longitudinal force. To take into account the influence of the remaining harmonics, one must retain the additional elements in the determinants of Eqs. (2. 22) and 2. 23).

As an example, we shall investigate the case of the piecewise-constant law of variation of the longitudinal force (• 5, No. 4). In this case

$$\mu_k = \frac{4\mu}{\pi k} \qquad (k = 1,\ 3,\ 5,\ \ldots),$$

and Eq. (2. 25) gives

$$\mu > \frac{k\Delta}{4} \qquad (k = 1,\ 3,\ 5,\ \ldots)$$

It is easily seen that in the case of piecewise-constant longitudinal force, the danger of secondary resonances is somewhat greater. Thus at a decrement of damping $\Delta = 0.01$, the third resonance can occur with a coefficient of excitation

$$\mu = \frac{3 \cdot 0{,}01}{4} = 0{,}0075$$

(instead of $\mu = 0.253$ in the case of a harmonic longitudinal force).

In the example investigated, the boundaries of the regions of instability can be determined with the same accuracy by means of the application of the criteria mentioned in • 7, No. 2. In fact, the characteristic roots in this case are found directly from Eq. (1. 19), where A is determined

according to Eq. (1.24). The equation for the computation of the boundaries of the regions of instability has the form

$$sT = \ln \left| A \pm \sqrt{A^2 - 1} \right|,$$

where

$$A = \left| \cos \frac{\pi p_1}{\theta} \cos \frac{\pi p_2}{\theta} - \frac{p_1^2 + p_2^2}{2 p_1 p_2} \sin \frac{\pi p_1}{\theta} \sin \frac{\pi p_2}{\theta} \right|.$$

REFERENCES

1. Rayleigh, Lord. The Theory of Sound. (The Macmillian Co., Ltd., London, 1926), 2nd ed.

2. Rzhanitsyn, A. R. The Stability of Equilibrium of Elastic Systems. (Ustolchivost ravnovesiya uprugikh sistem), (Gostekhizdat, Moscow, 1955).

3. Beilin, E. A. and G. U. Dzhanelidze. "Survey of Work on the Dynamic Stability of Elastic Systems." Priklad. Matem. i Mekh. 16 (5), 635-648 (1952). (Available in English as ASTIA No. AD-264148.)

4. Beliaev, N. M. "Stability of Prismatic Rods Subject to Variable Longitudinal Forces." (Collection), Engineering Construction and Structural Mechanics. (Inzhenernye sooruzheniya i stroitel'naya mekhanika), (Put', Leningrad, 1924), pp. 149-167.

5. Krylov, N. M. and N. N. Bogoliubov. "An Investigation of the Appearance of Resonance of the Transverse Vibrations of Rods Due to the Action of Normal Periodic Forces on an End." (Collection), An Investigation of Vibrations of Structures. (Issledovaniie kolebanii konstruktsii), (ONTI, Karkov-Kiev, 1935), pp. 5-24.

6. Kochin, N. E. "On the Torsional Vibrations of Crank Shafts." Priklad, Matem. i Mekh. 2 (1), 3-28 (1934).

7. Timoshenko, S. P. Vibration Problems in Engineering. (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1955), 3rd ed.

8. Bondarenko, G. V. The Hill Differential Equation and its Uses in Engineering Vibration Problems. (Uravnenye Khilla i evo primenenye v oblasti technicheskikh kolebanii), (Akad. Nauk SSSR, Moscow, 1936).

9. Mettler, E. "Biegeschwingungen eines Stabes unter pulsierender Axiallast." Mitt. Forsch.-Anst. GHH-Konzern 8, 1-12 (1940).

10. Utida, I. and K. Sezawa. "Dynamical Stability of a Column Under Periodic Longitudinal Forces." Rep. Aeronaut. Res. Inst. Tokyo Imp. Univ. 15, 139-183 (1940).

11. Lubkin, S. and J. J. Stoker. "Stability of Columns and Strings Under Periodically Varying Forces." Quart. Appl. Math. 1, 215-236 (1943).

12. Bodner, V. A. "The Stability of Plates Subjected to Longitudinal Forces." Priklad. Matem. i Mekh. 2 (N. S.) 87-104 (1938).

13. Khalilov, Z. I. "The Dynamic Stability of a Plate Under the Action of Periodic Longitudinal Forces." Trudy Anserv. gos. univ., ser. mat. 1, 28-32 (1942).

14. Einaudi, R. "Sulle Configurazioni di Equilibrio Instabile di una Piastra Sollecitata da Sforzi Tangenziali Pulsanti. Atti Accad. Gioenia 1, Memoria XX, 1-5 (1935/1936).

15. Ambartsumian, S. A. and A. A. Khachatrian. "On the Stability and Vibrations of Anisotropic Plates." Izvest. Akad. Nauk SSSR, Otdel. Tekh. Nauk (Mech. i mach.) 1, 113-122 (1960).

16. Dzhanelidze, G. Yu. and M. A. Radtsig. "Dynamic Stability of Rings Subject to Normal Periodic Forces." Priklad. Matam. i Mekh. 4 (N.S.), (5-6), 55-60 (1940).

17. Chelomei, V. N. The Dynamic Stability of Elements of Aircraft Structures. (Dinamicheskaya ustoichivost' elementov aviatsionuckh konstruktsii), (Aeroflot, Moscow. 1939).

18. Bolotin, V. V. "Dynamic Stability of Symmetric Modes of Oscillation of Compressed and Bent Arches." Doklady Akad. Nauk SSSR 83 (4), 537-539 (1952).

19. Bolotin, V. V. "Dynamic Stability of Plane Bending Modes." Inzheneryni Sbornik 14, 109-122 (1953).

20. Markov, A. N. "The Dynamic Stability of Anisotropic Cylindrical Shells." Priklad, Matem. i Mekh. 13, 145-150 (1949). (English translation available as Research and Development Technical Memo No. 44, Douglas Aircraft Company, Structures Section, Santa Monica California, 17 March 1961.)

21. Oniashvili, O. D. "On the Dynamic Stability of Shallow Shells." Soobshcheniya Akad. Nauk Gruzin SSR 9, 169-175 (1950).

22. Oniashvili, O. D. Certain Dynamic Problems of the Theory of Shells. (M. D. Friedman, Inc., West Newton, Mass., 1951) (Translated from Russian.)

23. Bolotin, V. V. "The Stability of a Thin Walled Spherical Shell Under the Action of a Periodic Pressure." (Collection), Calculations on Stability. (Rashchety na protsnost," (Mashgis, Moscow, 1958), 2, p. 284-289.

24*. Federhofer, Karl. "Die durch pulsierende Axialkräfte gedrückte kreiszylindershale." Osterr. Akad. Wiss., Math. -naturw. Kl. 163, 41-54 (1954). (Also available in English as TDR-169(3560-30)TR-3, Aerospace Corporation, El Segundo, California.)

25*. Yao, J. C. "Dynamic Stability of Cylindrical Shells Under Static and Periodic Axial and Radial Loads." TDR-169(3560-30)TR-1, Aerospace Corporation, El Segundo, California, (31 July 1962).

26. Bublik, B. M. and V. I. Merkulov. "On the Dynamic Stability of a Thin Elastic Shell Which is Filled with a Fluid." Priklad. Matem. i Mekh. 24, 941-947 (1960).

27. Mettler, E. "Biegeschwingungen eines Stabes mit Kleiner Vorkrümmung, exzentrisch angreifender pulsierender Axiallast und statischer Querbelastung." Forshungshefte aus Geb. des Stahlbaues 4, 1-23 (1941).

28. Naumov, K. A. "The Stability of Prismatic Rods Allowing for Damping." Trudy Moskov. Inst. Inzh. Shel. -dor. Transp. 69, 132-141 (1946).

29. Andronov, A. A. and M. A. Leontovich. "On the Vibrations of Systems with Periodically Varying Parameters." Zhur. Russ. Fiz. -Khim. Obshchestva 59, 429-443 (1927).

30. Smirnov, A. F. The Static and Dynamic Stability of Structures. (Staticheskaya i dinamicheskaya ustoichivost' sooruzhenii), (Transzheldorizdat, Moscow, 1947).

31. Makushin, V. M. "The Dynamic Stability of the Deformed State of Elastic Rods." Trudy Moskov. vissch. Tekh. utsch. im. Baumana (Chairman, Department of Strength of Materials, Div. III), 61-84 (1947).

*Reference numbers followed by asterisk have been added by translator.

32. Brachkovski, B. Z. "On the Dynamic Stability of Elastic Systems." Priklad. Matem. i Mekh. 6 (1), 87-88 (1942).

33. Bolotin, V. V. "On the Parametric Excitation of Transverse Vibrations." (Collection), Transverse Oscillations and Critical Rates. (Poperchnye kolebaniia i kriticheskie skorosti), (Akad. Nauk. SSSR, Moscow, 1953), p. 5-44.

34. Dzhanelidze, G. U. "Theorems on the Separation of Variables in Problems of the Dynamic Stability of Rods." Trudy Leningrad. Inst. Inzhen. vodnovo Transp. 20, 193-198 (1953).

35. Malkina, R. L. "The Stability of Curved Arches Subject to Longitudinal Periodic Foces." Inzhenernyi Sbornik 14, 123-130 (1952).

36. Salion, V. U. "The Dynamic Stability in Plane Bending of an I-Beam." Doklady Akad. Nauk SSSR (5) 375-381 (1950).

37. Salion, V. U. "The Dynamic Stability in Plane Bending." Doklady Akad. Nauk SSSR 78 (5), 873-875 (1951).

38. Salion, V. U. "The Dynamic Stability of a Curved Arch Under the Action of Periodic Moments (Non-Plane Deformations)." (Collection), Studies in Problems of Stability and Strength. (Issledovaniia po voprosam ustoichivosti i prochnosti), (Akad. Nauk Ukr. SSR, Kiev, 1956), p. 123-127.

39. Gol'denblat, I. I. Contemporary Problems of Vibrations and Stability of Engineering Structures. (Sovremennye problemy kolebanii i ustoichivosti inzhenernykh sooruzhenii), (Stroiizdat, Moscow, 1947).

40. Artem'ev, N. A. "Une méthode pour déterminer les exposants caractéristiques et son application à deux problèmes de la mécanique céleste." Izvest. Akad. Nauk SSSR ser Matem. 8, 61-100 (1944) (French Summary.)

41. Mettler, E. "Eine Theorie der Stabilitat der elastischen Bewegung," Ing. -Arch. 16. 135-146 (1947).

42. Weidenhammer, F. "Der eingespannte, axial-pulsierend belastete Stab als Stabilitätsproblem." Ing. -Arch. 19, 162-191 (1951).

43. Kucharski, W. "Beiträge zur Theorie der durch gleichförmigen schub beanspruchten Platte." Ing. -Arch. 18, I. Mitt., 385-393; II. Mitt., 394-408 (1950).

44. Reckling, K. A. "Die Stabilität erzwungener harmonischer Schwingungen gerader I-Träger im Verband eines Tragwerkes." Ing.-Arch. 20, 137-162 (1952).

45. Reckling, K. A. "Die dünne Kreisplatte mit pulsierender Randbelastung in ihrer Mittelebene als Stabilitätsproblem." Ing. Arch. 21, 141-147 (1953).

46. Reckling, K. A. "Die Stabilität erzwungener Schwingungen von Stäben and Trägern nach Rechnung und Versuch." Schiffstechnik 3, 135-140 (1955/1956).

47. Yakubovich, V. A. "An Observation on Some Works Concerning Systems of Linear Differential Equations with Periodic Coefficients." Priklad. Matem. i Mekh. 21, 707-713 (1957).

48. Yakubovich, V. A. "On the Dynamic Stability of Elastic Systems." Doklady Akad. Nauk SSSR 121, 602-605 (1958).

49. Yakubovich, V. A. "The Method of the Small Parameter for Canonical Systems with Periodic Coefficients." Priklad. Matem. i Mekh. 23, 15-35 (1959).

50. Piszczek, K. "Dynamic Stability of the Plane Form of Bending with Various Boundary Conditions." Rozpravy. Inż. 4, 175-225 (1956) (Polish with English and Russian summary.)

51. Piszczek, K. "Influence of Geometric and Dynamic Constraints on Resonance Regions in the Problem of the Dynamic Stability of a Thin-Walled Bar with Open Cross Sections." Rozpravy. Inż. 5, 207-227.

52. Piszczek, K. "The Influence of the Curvature of an Originally Curved Bar on the Resonance Regions of the Plane Form of Bending." Arch. Mech. Stosowanie 9, 155-189 (1957) (Polish and Russian summary).

53. Piszczek, K. "Parametric Combination Resonance (of the second kind) in Nonlinear Systems." Rozpravy. Inż. 8, 211-229 (1960). (Polish with English summary).

54. Bolotin, V. V. "Approximate Methods for Vibration Analysis of Frames." Trüdy Moskov. Energet. Inst. 17, 7-22 (1955).

55. Gol'denblat, I. I. Dynamic Stability of Structures. (Dinamicheskaya ustoichivost' sooruzhenii), (Stroiizdat, Moscow, 1948).

56. Mandel'shtam, L. I. and N. D. Papaleksi. "On the Establishment of Vibrations According to a Resonance of the nth Form." Zhur. Eksp. i Teoret. Fiz. 4, 67-77 (1934).

57. Bolotin, V. V. "On the Transverse Vibrations of Rods Excited by Periodic Longitudinal Forces." (Collection), 1 Transverse Oscillations and Critical Velocities. (Poperchnye kolebaniia i kriticheskie skorosti), (Akad, Nauk SSSR, Moscow, 1951).

58. Weidenhammer, F. "Nichtlineare Biegeschwingungen des axial-pulsierend belasteten Stabes." Ing.-Arch. 20, 315-330 (1952).

59. Bolotin, V. V. "On Errors in Certain Papers on Dynamic Stability." Izvest. Akad. Nauk SSSR Otdel. Tekh. Nauk 11, 144-147 (1955).

60 Mettler E. and F. Weidenhammer. "Der axial pulsierend belastete Stab mit Endmasse." Z. angew. Mathem. Mech. 36, 284-287 (1956).

61. Bolotin, V. V. "Determination of the Amplitudes of Transverse Vibrations Excited by Longitudinal Forces." (Collection), 2 Transverse Oscillations and Critical Velocities. (Poperchnye Kolebaniia i kriticheskie skorosti), (Akad. Nauk SSR, Moscow, 1953).

62. Bolotin, V. V. "On the Interaction of Forces and Excited Vibrations." Izvest. Akad. Nauk SSSR Otdel. Tekh. Nauk 4, 3-15 (1956).

63. Bolotin, V. V. "On the Mechanical Model Which Describes the Interaction of Parametrically Excited and Forced Vibrations." Trudy Moscov. Energet. Inst. 32, 54-66 (1959).

64. Ivovich, V. A. "On the Forced Pseudoharmonic Vibrations of Elastically Supported Bars." Doklady Akad. Nauk SSSR 119, 42-45 (1958).

65. Ivovich, V. A. "On the Sub-Harmonic Vibrations of Bars with Nonlinear Inertia." Doklady Akad. Nauk SSSR 119, 237-240 (1958).

66. Ivovich, V. A. "On the Nonlinear Bending Vibrations of Bars." Nauch. Doklady Visshiya Schkoly, ser. "Mashinostroienie i priborostroienie" 1, 96-102 (1958).

67. Ivovich, V. A. "Certain Nonlinear Problems of the Vibrations of Rods." Problemy Protshnosti v Machin. 5, 47-69 (1959).

68. Bolotin, V. V. "On the Bending Vibrations of Beams Whose Cross Sections have Different Principal Stiffnesses." Inzhenernyi Sbornik 19, 37-54 (1954).

69. Bolotin, V. V. "Certain Nonlinear Problems of the Dynamic Stability of Plates." Izvest. Akad. Nauk SSSR Otdel. Tekh. Nauk 10, 47-59 (1954).

70. Bolotin, V. V. "Parametric Excitation of Axisymmetric Vibrations of Elastic Arches." Inzhenernyi Sbornik 15, 83-88 (1953).

71. Burnashev, I. A. "On the Dynamic Stability of the Plane Bending Modes of a Beam." Doklady Akad. Nauk Uzbek. SSSR 3, 7-12 (1954).

72. Sobolev, V. A. "The Dynamic Stability of Deformation of a Strip in Excentric Compression and Pure Bending." Inzhenernyi Sbornik 19, 65-72 (1954).

73. Strutt, M. J. O. Lamésche Mathieusche und verwandte Funktionen in Physik und Technik. (Springer, Berlin, 1932).

74. McLachlan, N. W. Theory and Application of Mathieu Functions. (Oxford University Press, New York, 1947).

75. Meixner, J. and F. W. Schäfke. Matieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf physikalische und technische Probleme. (Springer-Verlag, Berlin, 1954).

76. Den Hartog, J. P. Mechanical Vibrations. McGraw-Hill Book Co., New York, 1956.)

77. Whittaker, E. T. and G. N. Watson. A Course of Modern Analysis. (University Press, Cambridge, 1952), 4th ed.

78. Mandel'shtam, L. I. and N. D. Papaleski. "Systems with Periodic Coefficients with Small Degree of Freedom and with Small Nonlinearities." Zhur. Eksp. i Teoret. Fiz. 15, 605-612 (1945).

79. Lazarev, V. A. "On Heteroparametric Excitation." Zhur. Tekh. Fiz. 4, 30-48 (1934).

80*. Weingarten, V. I. "Dynamic Instability of a Rod Under Periodic Longitudinal Forces." (To be published as a TDR series report, Aerospace Corporation, El Segundo, California.)

*Reference numbers followed by asterisk have been added by translator.

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Aerospace Corporation, El Segundo, California. THE DYNAMIC STABILITY OF ELASTIC SYSTEMS, VOLUME I, by V. V. Bolotin, trans. by V. I. Weingarten, K. N. Trirogoff, and K. D. Gallegos. 1 November 1962. [86] p. incl. illus. (Report TDR-169(3560-30)TR-2;SSD-TDR-62-154) (Contract AF 04(695-169) Unclassified report

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